ELEMENTARY STATICS

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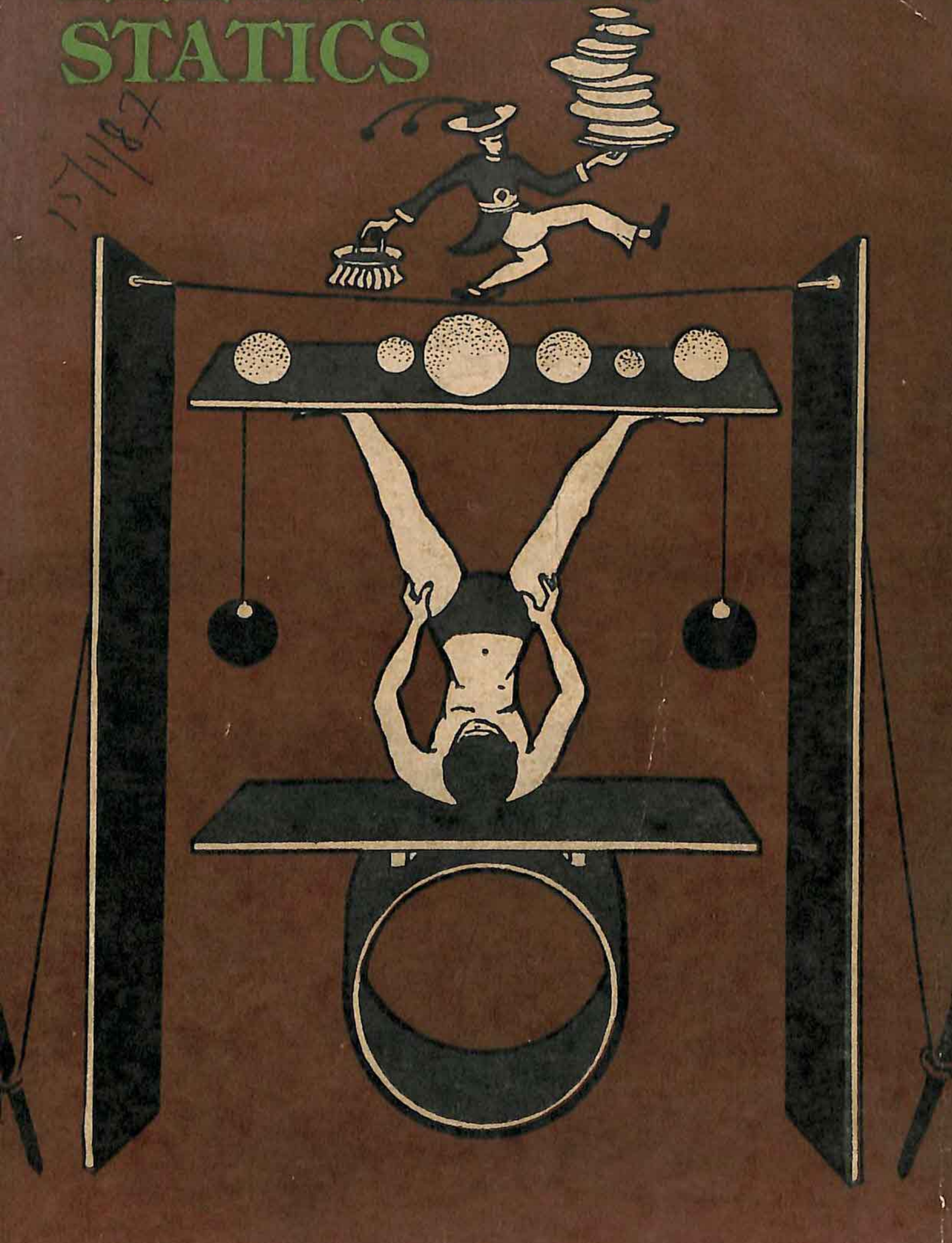

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ELEMENTARY STATICS

Author

S.C. DAS

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NCERT

National Council of Educational Research and Training

First Edition
April 1986

P.D. 5T

S.C.E.R.T., West Bengal
Date 14-1-87
Acc. No. 37157

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Rs. 30.30

Published at the Publication Department by the Secretary, National Council of Educational Research and Training, NIE Campus, Sri Aurobindo Marg, New Delhi-110016 and Printed by Mayur Press, 27 D.L.F. Industrial Area, Najafgarh Road, New Delhi-110015

Foreword

Students of secondary and senior secondary classes have generally to study a limited content in different subjects. Many bright and gifted students do not find the content to be challenging or rich enough to provide them the intellectual stimulation. This is true of mathematics as of other subjects. For instance, very limited content is available in the subject of mechanics which is generally included in physics and not in mathematics. It is felt that the limited knowledge in such a valuable subject is not sufficient to facilitate the bright student's pursuit of higher studies in Engineering, in Applied Mathematics or in any other fields where knowledge of mechanics is widely used. Further, this is an area in which even a science student can find wide application.

The present volume is the first of the two books which have been planned on elementary statics and dynamics. The second book on dynamics is under preparation by the same author. This book has been written with a view to meeting primarily the needs of the students studying at the plus two stage. Those preparing for many all-India competitive examinations may also find it useful.

The volume, although it contains largely enrichment materials, begins with very elementary topics in mechanics. A number of problems, easy and difficult and of various types, have been included to enable students to prepare for various examinations, competitive and others.

The necessity of producing such a volume for bright students has been felt for some time. NCERT, therefore, decided to undertake the production of a volume on such an important subject which can meet the needs of science students enrolled at the plus two stage.

The book represents a commendable effort on the part of Dr. S.C. Das, Professor in Mathematics in NCERT's Department of Education in Science and Mathematics. I am very glad to record my appreciation of the valuable contribution that Dr. Das has made through this book.

The Council will appreciate suggestions which may help to further improve the contents and presentation of the book.

P. L. MALHOTRA
Director
National Council of Educational
Research and Training.

New Delhi
April, 1986

Preface

The present book which is the first of my two books on Statics and Dynamics is intended for the students of senior secondary classes. Although Statics and Dynamics are included in the Mathematics syllabi of many States, these are not included in the existing NCERT's syllabus of Mathematics at plus two stage. It is true that students learn some portions of Mechanics consisting of Statics and Dynamics in books of Physics, but the scope of discussing various topics with different types of problems is very much limited in books of Physics. The importance of these two topics for a student of Mathematics is well-known to all. Mechanics is perhaps the only subject in the school, where a student can find the real application of most of the topics in mathematics like Trigonometry, Geometry, Algebra, Calculus and so on which are all taught in Mathematics in schools. If Mathematics is taught in the schools without showing its sufficient application in other branches of Science, then it becomes a dull subject to many students. Besides, in many all-India Competitive Examinations, particularly in the test papers for admission to various Engineering Colleges, challenging problems from these topics are asked. The performance of the students not learning these topics in Mathematics in schools becomes miserable in these examinations. The preparation of supplementary materials on these and similar other topics was, therefore, felt by many since long for the benefit of all science students at plus two stage. A project was, therefore, taken up by the Department for the preparation of these supplementary books on Mathematics. The second supplementary book on Dynamics which is in preparation is expected to be printed soon.

All the topics which are generally included in most of the elementary books on Statics have not been included in this book for the obvious reason of reducing the bulkness of the book. Only those topics which are included in the syllabi of most of the States and various all-India Competitive Examinations have been selected for this book and emphasis has been given on the basic theory as well as various types of challenging problems.

The book has been started with the explanation of the basic concept in physical quantities. Knowledge of Mathematics beyond plus two stage has not been assumed anywhere in this book. A large number of problems, simple and difficult, have been worked out and given in the exercises so as to enable all students, gifted or average, to use the book without much difficulty. The S.I. system of all physical quantities has been used throughout this book. Some well-known problems which were asked in various examinations have been included in this book with slight modification by changing the system of units and removing the vagueness in the traditional language. A number of multiple-choice objective type questions from each topic are given in the appendix to facilitate students appearing at various all-India Competitive examinations. It is hoped that such students as well as students of physics may be benefited from this book.

The entire manuscripts prepared by me, have been reviewed by eminent experts in Applied Mathematics. They are Dr. A. Ramakrishnan, Anna University, Madras, Dr. Rasojit Bera, Bengal Engineering College, West Bengal, Prof. S. C. Das, Birla Institute of Technology, Ranchi, Dr. A. Satyanarayana Rao, Andhra University, Waltair, Dr. S. S. Ray, Visva-Bharati University, Shantiniketan, West Bengal, Dr. B. S. Goel, formerly in Regional College of Education, Mysore and Sri H. Ghosh, Bengal Engineering College, West Bengal. I am grateful to all of them. The manuscripts were again revised thoroughly by me in the light of the comments of the reviewers. I am also thankful to Dr. R. Bera of B. E. College, W. B. and Sri S. K. Mukherjee of D. M. College, Imphal for their contribution in framing some of the multiple-choice objective type questions which have been revised and included in the appendix. My thanks are also due to Shri A. Chakraborty, Lecturer, A.V., NCERT, who visualised the design of the cover of this book.

Although maximum efforts have been made in detecting and removing the errors in the book, I shall be very grateful to any reader who may bring out error, if any, to my notice. Suggestions from the readers for further improvement of the book will be highly appreciated.

S.C. DAS
Author

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Chapter I

Some fundamental concepts

1.1 Definitions

Matter : Any substance in this universe, which occupies space and can be perceived by our sense is called a matter.

Body : A body is a portion of matter bounded in all directions. It, therefore, occupies a limited space and has a finite volume. Two bodies occupying equal amounts of space may contain unequal quantities of matter due to unequal distribution of matter in the two bodies. The units of mass are gramme, pound and kilogramme in C.G.S., F.P.S. and M.K.S. systems respectively.

Mass : It is the quantity of matter contained in a body. The greater the mass of a body, the more it has the power of offering resistance to motion.

Particle : It is a very small portion of matter whose dimensions are negligible. It may be represented by a geometrical point having concentrated mass. The concept of a particle is idealistic. Whenever the dimensions of a body are negligible compared to the neighbouring objects but its mass is of appreciable magnitude, the body may be considered as a particle possessing position and mass only.

1.2 Definition of Statics and related branches of Science

Before giving the definition of Statics and other related branches of science, we first difine a force.

Force : The formal definition of a force is given by Newton's first law of motion, which states as follows : *Every body continues in its state of rest or of unifom motion in a straight line except in so far as it is compelled by external impressed force to change that state.*

This law can be divided into three parts. The first part deals with the state of rest of a body. It states that a body can not change its position on its own accord. It has no power within itself to change its own state of rest. The second part deals with the state of uniform motion in a straight line i.e. when a body tends to move due to the action of some other body with a definite velocity in a definite direction, it will move with the same velocity in the same direction for ever provided there is no cause to interfere with this subsequent motion. It has no power within itself to change its own state of uniform motion in a straight line. The third part of the law gives the definition of a force. It states that wherever there is a change of the state of rest or of uniform motion in a straight line or a tendency to either of such changes, it implies that the body is acted upon by some other body. Thus the formal definition of a force is as follows :

A force is an action exerted upon a body in order to change its state either of rest or of moving uniformly forward in a straight line.

Statics : It is a branch of *Mechanics* which is the science dealing with the action of forces on bodies. The bodies which are acted on by a number of forces may either be in motion or remain at rest. The part of Mechanics which deals with the motion of solid bodies is called *Dynamics* and that which deals with the motion of liquids or gases is called *Hydrodynamics* The part of mechanics which deals with action of forces on a body or a system of bodies at rest is called *Statics* when the bodies are solids and *Hydrostatics* when the bodies are liquids or gases. Here in this book, we shall be concerned with Statics only.

Remark : Statics is, in foct, a particular case of Dynamics in which the motions of the bodies are zeros. The parallelogram law of forces, which is the fundamental theorem of Statics, is based on the fundamental laws of Dynamics, known as Newton's laws of motion. Many results in connection with equilibrium of forces are obtained from Dynamics by supposing that motion takes place to a very small extent.

1.3 Some more definitions

Bodies at rest and in motion :

A body is said to be in motion if its position with respect to surrounding objects is changing. There are mainly two types of motion : rotatory and translatory. In a translatory motion, a body moves in such a manner that all the particles constituting it have exactly the same change of position in the same direction, so that any line in the body remains parallel to itself throughout the motion. In a rotatory motion, a body moves round a fixed point or a fixed axis.

A body is said to be *at rest* or *in equilibrium* when it does not change its position relative to surrounding objects. When a body is at rest, both the translatory and rotatory motions are absent.

Remark : Strictly speaking, there is no body on this earth which is absolutely at rest, because the earth itself is moving. All bodies on the earth which are at rest relative to the earth should, therefore, be called "bodies at rest relative to the earth". However, in this book we shall use the phrase "bodies at rest or in equilibrium" to mean that they are at rest relative to the earth. But when a man is sitting on a running train, we shall say that the man is at rest relative to the train" and he is moving with the speed of the train relative to a stationary object, say a tree, outside the train.

Remark 2 : A particle is assumed to have no rotatory motion which a body possesses. It can have merely a motion of translation.

Rigid body : It is a body whose size and shape are supposed to be unaltered even when it is acted on by any forces whatsoever and the distance between any two points of it is invariable.

It may be mentioned here that there is no body in nature which is perfectly rigid i.e. which is not deformed when it is subjected to forces. However hard bodies may be, these are more or less elastic and subject to extension or distortion under the action of forces. Hence, treating such bodies as perfectly rigid is an ideal conception and untenable. But in Statics, we shall assume that all bodies are *perfectly rigid*, because this assumption is necessary for building up elementary principles in Statics.

Forces in equilibrium :

If a number of forces act on a body or a particle and keep it at rest, then the forces are said to be in equilibrium.

Equal forces : Two forces are said to be equal when their magnitudes are the same and they act in the same direction.

Smooth bodies : Two bodies are said to be smooth if, when they press against one another, the forces acting along their common surface are negligible. In fact, there is no such thing as a perfectly smooth body. But in solving many problems in Statics, the two bodies pressing each other are supposed to be perfectly smooth.

Resultant force and components :

If two or more forces act on a particle or on a body at a point in different directions and if a single force can be found out such that the effect of this single force is the same as the joint effect of all the forces, then the single force is called the *resultant* of the system of forces and the system of forces is called the *components* of the resultant.

1.4 Characteristics of a force

A force acting on a body is specified copmletely when it possesses the following characteristics :

(i) Its point of application.
(ii) Its magnitude.
(iii) Its direction or sense.
(iv) Its line of action.

The point of application of a force is the point of the body at which the force acts. Let A be the point of application. The magnitude of the force is represented on some suitable scale by the length of the line segment, say AB. If the force acts from A to B, its direction or sense is indicated by putting an arrow on the line from A to B. The line of action of the force is the line AB along which the force acts. This force can be written in vector notation as $\overrightarrow{AB}$. If the line of action has also to be specified along with the magnitude and direction it can be treated as a localised vector relative to a frame of reference in the body.

1.5 Different kinds of forces

In Statics, we shall deal with three different kinds of forces.

(i) *Attraction and repulsion: Weight*

Attraction and repulsion are forces which act between two unconnected bodies. When the two bodies tend to approach each other due to force, the force is called *attraction* and when they tend to separate out due to the force, the force is called *repulsion*. According to the law of gravitation, any two bodies attract each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The earth attracts all bodies on its surface with a force known as gravity. We also find the force of attraction between two unlike magnetic poles and force of repulsion between two like magnetic poles.

The *weight* of a body is the force with which it is attracted towards the centre of the earth. Its direction is always assumed to be vertical. The difference between the weight and mass of a body should be clearly understood. The weight of a body depends on the attraction of the earth, but the mass of a body does not depend on it. If a body is taken to the centre of the earth or to a place where there is no attraction, the body will have no weight, but its mass will always remains the same. As the attraction of the earth varies from place to place, the weigth of a body will also vary from place to place, but the mass of a body, which depends on the quantity of matter in the body remains the same at all places on the earth. We compare two masses by a common balance bul we measure a force by a spring balance.

(ii) *Action and Reaction* :

These forces are produced when two bodies are in direct contact with each other. When two bodies are in actual contact with each other, each will exert a force on the other. If the force exerted by any one of the two bodies upon the second body be called action, then the force exerted by the second body on the first will be called reaction. By Newton's third law of motion which states that "to every action, there is an equal and opposite reaction", these action and reaction are always equal in magnitude but opposite in sign. Thus, if two bodies A and B are in contact and the body A exerts a force R on the body B, then B will also exert an equal force R on A but in the opposite direction (fig. 1). For example, if you put a book on a table, the book does not penetrate into the table, beacause the action and reaction between the table and the book are equal and opposite. Similarly, you can stand on the floor only because the floor supports your weight by giving you an equal and opposite upward reaction.

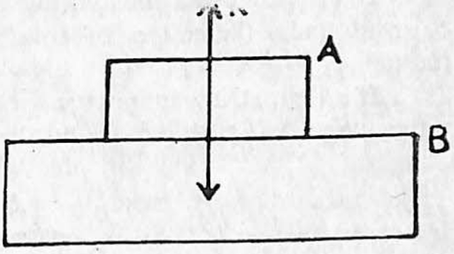


Fig. 1

Nomal reaction : When two bodies pressing against one another are supposed to be smooth, the reaction of one body on the other is always perpendicular to their common surface. This reaction is called the *normal reaction* between the two smooth bodies. There is no tangential force at the point of contact. But in the case of two rough bodies in contact, besides the nomal reaction between the two bodies, a tangential reaction called the force of friction acts at the point of contact. Reaction at the hinge or a joint in a framework will be discussed later at a proper place.

Remark (*a*) : When a rod rests against a smooth plane or a smooth peg, then the reaction of the plane or the peg on the rod will be perpendicular to the rod.

(*b*) When the end of a rod or a particle rests on the curved surface of a smooth sphere or against a smooth circular arc, then the reaction of the surface or the circular arc on the end of the rod or the particle will be normal to the sphere or the circle and so it will pass through its centre.

(iii) *Tension or pull of a string or a rod and thrust along a rod* :

When two particles or bodies are connected by a string and the particles or bodies are subjected to some forces which make the string stretched, then the fibres of the string are subject to a certain pulling action throughout its length (fig. 2). This pulling action is called the *tension* or *pull* of the string. If this tension is increased beyond a certain limit, then the string will break at a certain point. If we consider any point P on the string, then we can consider the part AP as the body on which the part BP exerts a force and the part BP as the body on which the part AP exerts a force. For equilibrium, these two forces must be equal in magnitude and act in the opposite directicns.

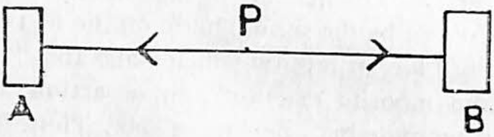


Fig. 2

In the case of an inextensible string i.e. a string having no elongation or strain under the action of forces, *the tension of the string is constant throughout its length* provided the weight of the string is negleced.

If a light string supports a weight and is passed over a small smooth pulley as at A (fig. 3), it is found that the force required to keep the weight in position is unaltered even if the string is held in the position AB, AC or AD, Hence, *the tension in a light string passing round a smooth peg or pulley is the same throughout its length.*

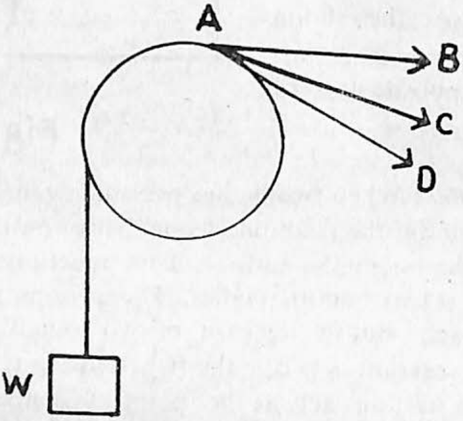


Fig. 3

It should be noted that if two or more strings are knotted together, the separate portions become distinct strings and the tensions in these portions are not necessarily the same. In other words, *tensions in different portions of a knotted string are, in general, different.*

The *thrust* along a rod can be explained exactly in the same way as a light string. In the case of a light rod, forces exerted along the rod may be outwards or inwards. When the force is outwards, it is called *tension of the rod* and when the force is inwards, it is called the *thrust along the rod.* When two persons push the two ends of a rod towards each other, the thrust is produced and the tension is generated when they pull the rod away from each other.

1.6 Units of force

The unit of a force is defined as that force which acting on a unit mass produces a unit acceleration. There are various systems in which the unit of a force can be expressed.

(i) centimetre-gramme-second (C.G.S.) System.
(ii) foot-pound-second (F.P.S.) System
(iii) metre-kilogramme-second (M.K.S.) System.

In C.G.S. System, the unit of force is called *dyne.* This is defined as the force which acting on a mass of one gramme, produces an acceleration of one centimetre per second per second. In F.P.S. System, the unit of force is called *poundal.* This is defined as the force which acting on a mass of one pound produces an acceleration of one foot per second per second. In M.K.S. System, the unit of force is called *newton.* This is defined as the force which acting on a mass of one kilogramme produces an acceleration of one metre per second per second. This is also called the S.I. System of force, which is now widely used in all branches of science. We shall use M.K.S. System in this book.

1.7 Axioms in Statics

There are three axioms in Statics, which follow from Newton's three laws of motion.

(i) *The principle of the inertness of matter* :

Newton's first law of motion, already stated, asserts this principle. This principle states that a body at rest will continue to remain at rest and a body in uniform motion along a straight line will continue to move along the straight line with the same velocity if there is no external force acting on the body This principle can be regarded as a matter of common experience. We notice that bodies at rest do not begin to move unless forced to do so by some external cause.

(ii) *Principle of physical independence of forces* :

This principle is asserted by Newton's second law of motion.

The rate of change of momentum of a body is proportional to the impressed force and takes place in the direction in which the force acts.

This principle states that the effect of a force on a body will be produced in its own direction under all circumstances i.e. the effect of a force is not affected by the presence of the other forces. Thus, if there are two systems of forces, each of which acting alone on a body keeps the body in equilibrium, the body will still be in equilibrium if both the systems act simultaneously on the body, since each system produces its own effect.

Hence, if a number of forces are in equilibrium, we can add or remove any particular forces which by themselves are in equilibrium; for example, two equal and opposite forces acting along the same straight line.

(iii) *Principle of transmissibility of a force* :

This principle is asserted by Newton's third law of motion already stated. This principle states that when a force acts at a point on a body, its effect is unaltered if its point of application is transferred to any other point on its line of action, provided the two point are rigidly connected with one another :

This principle follows as an immediate consequence of the following statical axiom :

"*Two forces acting at two points of a rigid body will keep it in equilibrium if and only if they are equal in magnitude and opposite in direction and act in the same straight line*".

It follows from the above principle that when a force acts on a body, we can take any point on its line of action as the point of application of the force.

Chapter II

Resultants and Components of Forces

2.1 Introduction

In chapter I, we have given the definition of a resultant and its components. The effecct of a number of forces acting on a particle is always the same as that of its resultant. Hence, whenever a system of forces acting on a body has a resultant, the system of forces can be replaced by their resultant. Conversely, any force acting on a body can be replaced by its components. It is, of course, not necessary that a system of forces acting on a *rigid body* will always have a single resultant. This will be explained in a subsequent chapter. But if a number of forces act on a *particle*, we can always find out their resultant which may be zero in some particular cases. In this chapter, we shall discuss how to find the resultant of a number of forces acting at a point. It should be noted here that whenever we speak of forces "acting at a point", it will be understood to mean that the forces actually act on a particle placed at that point.

We shall first consider the case of two forces acting at a point. If they act along the same straight line and in the same direction, then their resultant is obviously equal in magnitude to the sum of their magnitudes and acts in the same direction. If the forces act along the same straight line but in the opposite directions, then the magnitude of their resultant is equal to the difference of their magnitudes and acts in the direction in which the greater force acts. When the two forces act on a particle and are inclined to each other, then their resultant can be found with the help of a fundamental principle of Statics, known as the law of parallelogram of forces.

2.2 The parallelogram of forces

If two forces acting at a point on a body be represented in magnitude and direction by the two adjacent sides of parallelogram drawn from the point, then their resultant is represented in magnitude and direction by the piagonal of the parallelogram drawn from that point.

Although no satisfactory formal proof of the theorem can be given without introducing the idea of motion, the above theorem has been verified experimentally.

The above theorem states that if two forces of magnitudes P and Q act a point O of a body and if these two forces be represented on a chosen scale by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and if the parallelogram $OACB$ is completed with OA and OB as adjacent sides, then the resultant force, R, will be represented on the same scale by the diagonal OC drawn from O, in magnitude and direction (fig. 4).

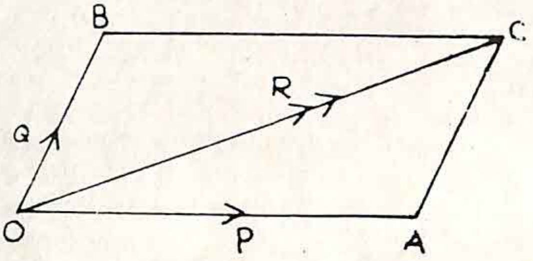
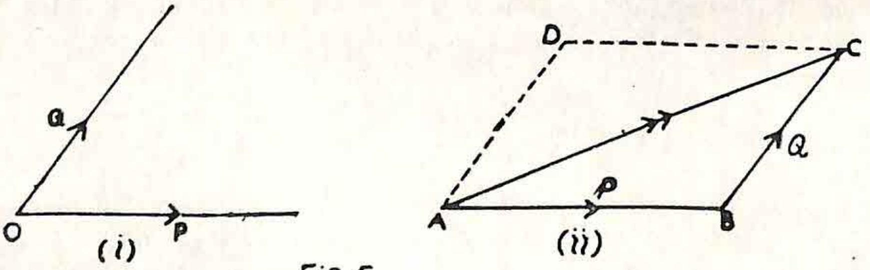


Fig. 4

It follows that if two forces, P and Q act at a point O such that they can be represented in magnitude and direction, by the two sides AB and BC of a $\triangle ABC$, taken in the same order, then the third side AC of the triangle taken in the opposite order will represent the resultant of the two forces in magnitude and direction only, but AC will not necessarily be the line of action of the resultant. The resultant force will actually act at O (fig. 5).



(i) (ii)

Fig. 5

This also follows from the vector law of addition :

$$\vec{AB}+\vec{BC}=\vec{AC}$$

It is seen that if we complete the parallelogram $ABCD$, then $\vec{AD}=\vec{BC}$, so that the force Q may also be reperesented by the side AD.

Hence, by parallelogram of forces, the diagonal AC represents the resultant of P and Q in magnitude and direction.

2.3 Analytical deduction for the resultant of two forces

Let P and Q be the magnitudes of the forces represented by $\vec{OA}$ and $\vec{OB}$ and let the forces act at the point O making an angle α with each other so that $\angle AOB=\alpha$. Complete the parallelogram $OACB$ with OA and OB as the adjacent sides. Then by the parallelogram law of forces, the diagonal vector $\vec{OC}$ represents the resultant of magnitude R, say. Let the resultant of magnitude R make an angle θ with the force of magnitude P so that $\angle AOB=\theta$.

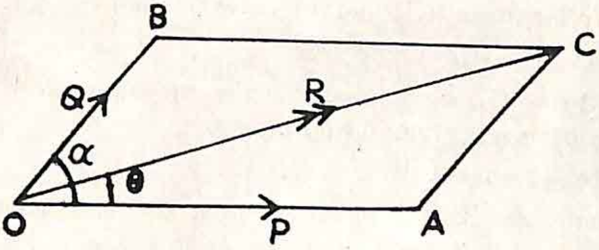


Fig. 6

Now, in triangle OAC, we have

$OA=P,\ AC=OB=Q,\ OC=R.$

Hence, using cosine formula in $\triangle OAC$, we have

$$OC^2=OA^2+AC^2-2OA.\ AC\cos\angle CAO$$

$$\Rightarrow R^2=P^2+Q^2-2PQ\cos(\pi-\alpha)$$

$$=P^2+Q^2+2PQ\cos\alpha$$

$$\Rightarrow R=\sqrt{P^2+Q^2+2PQ\cos\alpha} \qquad \ldots(1)$$

[taking the positive square root only, since we are concerned with the magnitude of the resultant only]

To find the direction θ of the resultant, we use the sine formula in $\triangle OAC$. Thus we have

$$\frac{AC}{\sin \angle AOC}=\frac{OA}{\sin \angle OCA}$$

$$\Rightarrow \frac{Q}{\sin\theta}=\frac{P}{\sin(\alpha-\theta)} \qquad [\because \angle ACB=\angle AOB=\alpha \text{ and } \angle OCB=\angle AOC=\theta]$$

$$\Rightarrow Q(\sin\alpha\cos\theta-\cos\alpha\sin\theta)=P\sin\theta$$

$$\Rightarrow (Q\cos\alpha+P)\sin\theta=\sin\alpha\cos\theta$$

$$\Rightarrow \tan\theta=\frac{Q\sin\alpha}{P+Q\cos\alpha}$$

$$\Rightarrow \theta=\tan^{-1}\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right) \qquad \ldots(2)$$

(1) and (2) give the magnitude and direction of the resultant.

Particular cases :

Case 1. From (1), we observe that the magnitude R of the resultant is greatest when $\cos\alpha$ is greatest i.e., when

$$\cos\alpha=1\Rightarrow\alpha=O.$$

Hence, the resultant is greatest when the component forces act along the same line in the same direction. In this case, $R=P+Q$.

Case 2. R is least when $\cos\alpha$ is least i.e. when

$$\cos\alpha=-1\Rightarrow\alpha=\pi.$$

Hence, the resultant is least when the component forces act along the same line but in the opposite directions. In this case,

$$R=P-Q, \text{ if } P>Q$$
$$=Q-p, \text{ if } Q>P.$$

The direction of the resultant will be in the direction of the greater force.

Case 3. If the component forces of magnitudes P and Q are perpendicular to each other, then $\alpha=\frac{\pi}{2}$.

In this case

$$R=\sqrt{P^2+Q^2}.$$

Case 4. If the component forces are of equal magnitude i.e. if $P=Q$, then

$$R=\sqrt{2}P\sqrt{1+\cos\alpha}=2P\cos\frac{\pi}{2}$$

and from (2),

$$\tan\theta=\frac{\sin\alpha}{1+\cos\alpha}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}=\tan\frac{\alpha}{2}$$

$$\Rightarrow \theta=\frac{\alpha}{2}.$$

Hence, the line of action of the resultant of two forces of equal magnitude bisects the angle between them.

Case 5. If $R=P=Q$, then from (1),

$$P=\sqrt{2P^2+2P^2\cos\alpha}$$

$$=\sqrt{2}P\sqrt{1+\cos\alpha}=2P\cos\frac{\alpha}{2}$$

$$\Rightarrow \cos\frac{\alpha}{2}=\frac{1}{2}\Rightarrow\frac{\alpha}{2}=60^\circ\Rightarrow\alpha=120^\circ.$$

Hence, if the angle between two forces of equal magnitude is 120°, then the magnitude of their resultant is equal to that of each of the component forces.

2.4 Resolution of a force into two components in two given directions

In the previous section, the formulae for the resultant of two given forces acting at a point were obtained. The converse problem of finding two forces acting at a point in two given directions such that their resultant is a given force at that point can be discussed now.

Let $\overrightarrow{OC}$ represent the given force of magnitude R, which makes angles α and β with the given directions OX and OY. It is required to find out the components of the force R along OX and OY. Complete the parallelogram $OACB$ with diagonal OC and two adjacent sides along OX and OY (fig. 7). Then by parallelogram of forces, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the components of the force $\overrightarrow{OC}$. Let the magnitudes of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be P and Q respectively, measured on the same scale as the magnitude of the force $\overrightarrow{OC}$ is represented by R.

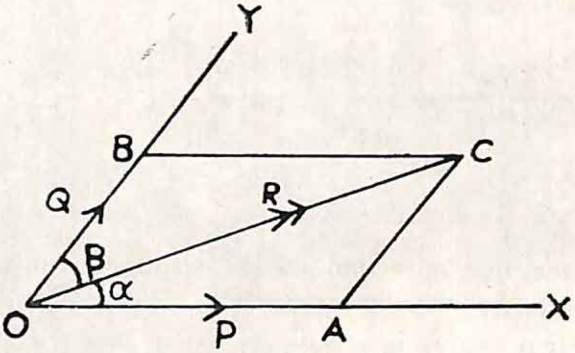


Fig. 7

Now, since $AC \parallel OB$, hence, $\angle OCA = \beta$.

From $\triangle OCA$, we have by sine formula,

$$\frac{OA}{\sin\beta} = \frac{AC}{\sin\alpha} = \frac{OC}{\sin[180° - (\alpha+\beta)]}$$

$$\Rightarrow \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha} = \frac{R}{\sin(\alpha+\beta)}$$

$$\therefore\ P = \frac{R\sin\beta}{\sin(\alpha+\beta)},\quad Q = \frac{R\sin\alpha}{\sin(\alpha+\beta)} \qquad \ldots(1)$$

Particular case :

If the two components of the resultant are perpendicular to each other, then $\alpha+\beta = \frac{\pi}{2}$.

In this case, each of the components is called the *resolved part* of the force in the corresponding direction.

Now, if $\alpha+\beta = \frac{\pi}{2}$, then from (1), we get

$$P = R\sin\beta = R\sin\left(\frac{\pi}{2} - \alpha\right) = R\cos\alpha$$

and $Q = R\sin\alpha$.

In other words, *the resolved part of a force of magnitude R along a line making an angle α with the line of action of the force is R cos α and that perpendicular to this line is R sin α (Fig. 8)*

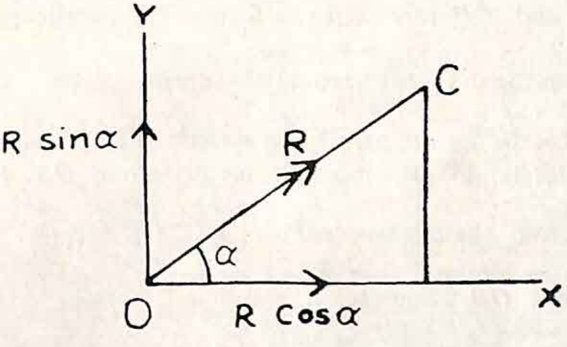


Fig. 8

It follows that the resolved part of a force of magnitude R perpendicular to it is $R \cos 90° = 0$. In other words, *a force of magnitude R produces no translatory effect in a direction perpendicular to it.*

Remark I: If two forces are given, we can find their resultant in a unique way. But the reverse is not true i.e. a single force can be split up into two compoments in an infinite number of ways, for an infinite number of parallelograms can be constructed wiih the segment representing the given force as the common diagonal. However, the components of the given force will be unique in two specified directions.

Remark 2: The resolved parts of the force $\overrightarrow{OC}$ in the perpendicular directions OX and OY are also given by $\overrightarrow{OC}.\vec{i}$ and $\overrightarrow{OC}.\vec{j}$ respectively where $\vec{i}$ and $\vec{j}$ are unit vectors in the directions OX and OY respectively.

Remark 3: The *resolved parts* of a force are defined with reference to two given directions which are perpendicular to each other while the *components* are defined with reference to any two given directions.

Remark 4: The resolved parts of a force in a pair of perpendicular directions are also called the *resolutes* of the force in these directions.

2.5 Relation between the resolved parts of the resultant and of its components

The relation between the resolved part of the resultant and the resolved parts of its components is given by the following theorem :

Theorem: The algebraic sum of the resolved parts of any two forces acting at a point, along any direction is equal to the resolved part of their resultant in the same direction.

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent two forces. The parallelogram $OACB$ is completed. Then the diagonal vector $\overrightarrow{OC}$ represents the resultant of the forces $\overrightarrow{OA}$ and $\overrightarrow{OB}$, by the parallelogram law of forces. Let us take any line OX through O and consider a unit vector i along OX. By remark of the previous section, the resolved parts of $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ along OX are respectively $\overrightarrow{OA}.\overrightarrow{i}$, $\overrightarrow{OB}.\overrightarrow{i}$ and $\overrightarrow{OC}\ \overrightarrow{i}$.

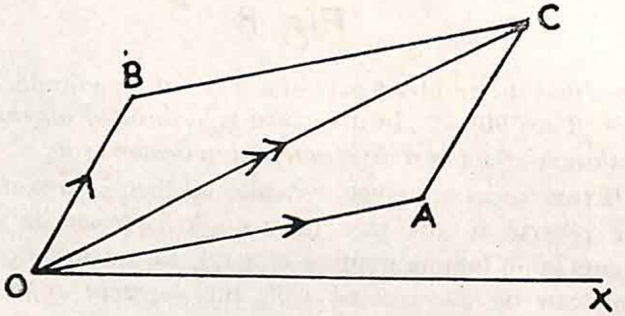


Fig. 9

Now, the sum of the resolved parts of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ along OX is

$$\overrightarrow{OA}.\overrightarrow{i}+\overrightarrow{OB}\ .\overrightarrow{i}$$

$$=(\overrightarrow{OA}+\overrightarrow{OB}).\ \overrightarrow{i}\ =\overrightarrow{OC}.\ \overrightarrow{i},$$

since by parallelogram law of forces, we have $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$.

Hence, the result follows.

By the repeated application of the above theorm, the generalised theorem stated below can be proved.

Generalised theorem :

The algebraic sum of the resolved parts of any number of forces acting at a point, in any direction is equal to the resolved part of their resultant in the same direction.

2.6 Resultant of any number of coplanar forces acting at a point

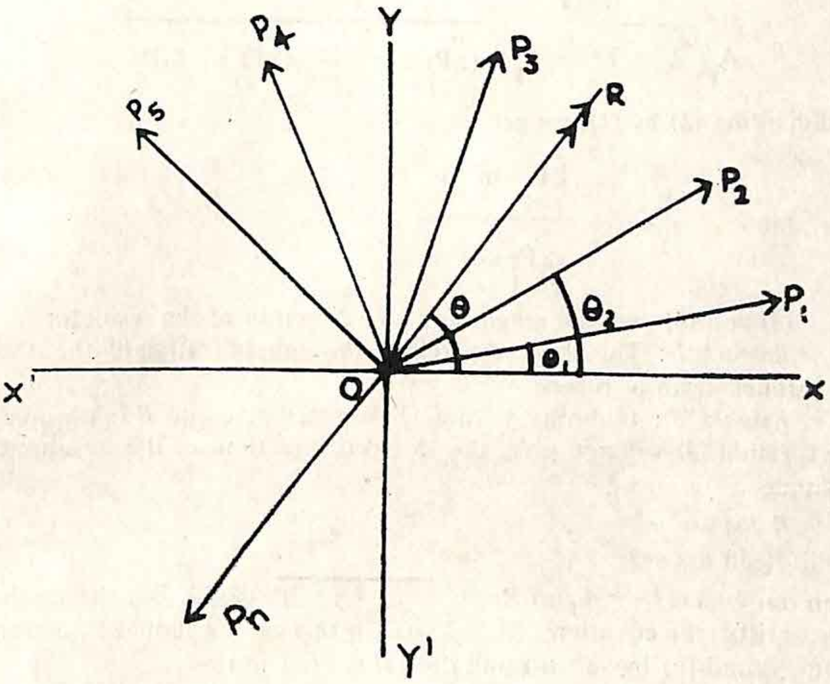


Fig. 10

Let n coplanar forces of magnitudes $P_1, P_2 \ldots\ldots P_n$ act at a point O and let R be their resultant. Let the forces of magnitudes $P_1, P_2, \ldots\ldots P_n$ and R make angles $\theta_1, \theta_2, \ldots\ldots \theta_n$ and θ with OX, OY being perpendicular to OX. Let X and Y be the algebraic sums of the resolved parts of the forces with magnitudes $P_1, P_2, \ldots\ldots P_n$ along X and Y axes respectively. Equating the algebraic sum of the resolved parts of the forces of magnitudes $P_1, P_2, \ldots\ldots P_n$ to the resolved part of the force of magnitude R along OX, we get
$X = P_1 \cos \theta_1 + P_2 \cos \theta_2 + \ldots + P_n \cos \theta_n = R \cos \theta$. ...(1)

Similarly, equating the algebraic sum of the resolved parts of the forces of magnitudes $P_1, P_2, \ldots P_n$ to the resolved part of the force of magnitude R along OY, we get

$Y = P_1 \sin \theta_1 + P_2 \sin \theta_2 + \ldots + P_n \sin \theta_n = R \sin \theta$ ...(2)

Squaring (1) and (2) and adding, we get

$$R^2=X^2+Y^2=\left(\sum_{r=1}^{n} P_r \cos\theta_r\right)^2+\left(\sum_{r=1}^{n} P_r \sin\theta_r\right)^2$$

$$R=\sqrt{X^2+Y^2}=\sqrt{\left(\sum_{r=1}^{n} Pr\cos\theta r\right)^2+\left(\sum_{r=1}^{n} Pr\sin\theta r\right)^2} \quad \text{.. (3)}$$

and dividing (2) by (1), we get

$$\tan\theta=\frac{Y}{X}=\frac{\sum_{r=1}^{n} Pr\sin\theta r}{\sum_{r=1}^{n} Pr\cos\theta r} \quad \text{. (4)}$$

(3) and (4) give the magnitude and direction of the resultant.

Remark 1 : The above theorem is the generalization of the theorem of parallelogram of forces.

Remark 2 : If both X and Y are negative and R is positive, then the formula (4) will not give the correct direction of the resultant. For example, if

$R\cos\theta=-3$ ...(i)

and $R\sin\theta=-3$ ...(ii)

then $\tan\theta=1\Rightarrow\theta=\pi/4$ and $R=\sqrt{(-3)^2+(-3)^2}=3\sqrt{2}$. But these values do not satisfy the equations (i) and (ii). In this case, θ should be determined from (i) and (ii) by substituting the value of R in it.

2.7 An important theorem

The resultant of two forces acting at a point O in the directions OA and OB and represented in magnitude by m. OA and n. OB respectively is represented in magnitude by (m+n). OC acting along OC, where C is a point on AB such that AC : CB = n : m.

Let two forces m $\overrightarrow{OA}$ and n $\overrightarrow{OB}$ act at O. To show that their resultant is $(m+n)\,\overrightarrow{OC}$.

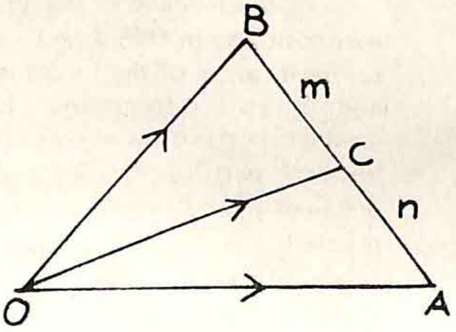


Fig. 11

By triangle law of vectors, we have

$$\vec{OA}=\vec{OC}+\vec{CA}$$

$$\Rightarrow m\,\vec{OA}=m\,\vec{OC}+m\,\vec{CA}. \quad \ldots(1)$$

Also $\vec{OB}=\vec{OC}+\vec{CB}$

$$\Rightarrow n\,\vec{OB}=n\,\vec{OC}+n\,\vec{CB} \quad \ldots(2)$$

Adding (1) and (2), we get

$$m\,\vec{OA}+n\,\vec{OB}=(m+n)\,\vec{OC}+m\,\vec{CA}+n\,\vec{CB} \quad \ldots(3)$$

Now, $\frac{AC}{CB}=\frac{n}{m}$

$$\Rightarrow m\,\vec{AC}=n\,\vec{CB}$$

$$\Rightarrow n\,\vec{CB}+m\,\vec{CA}=0.$$

Hence, from (3), it follows that

$$m\,\vec{OA}+n\,\vec{OB}=(m+n)\,\vec{OC}.$$

This proves the theorem.

Particular case :

If $m=n=1$, then we see that

$$\vec{OA}+\vec{OB}=2\,\vec{OC}$$

i.e. the resultant of two forces $\vec{OA}$ and $\vec{OB}$ is $2\,\vec{OC}$, where C is the middle point of AB,

2.8 A note on some conventional terms used in Statics

In any book of Mechanics, we shall find a few terms or statements some of which seem to be incorrect if interpreted rigorously and some may give vague ideas to students. These conventional terms or statements are used in books of Mechanics just to avoid too many repetitions of the same word or words and for convenience. They will appear to be incorrect if interpreted wrongly. For example, we generally say in short that "a force P acts at a point". By this statement, we actually mean that a force *of magnitude* P acts *on a particle placed* at a point. However, if we interpret strictly the scalar number P as a force which is a vector quantity, then the statement is definitely wrong. Henceforth, the same conventional terms and statements will be used in this book also for simplicity. But the actual meaning of these terms and statements should not be misunderstood by students. They should be very clear about the actual correct meaning of these terms and statements. Below is given the list of a few such conventional terms and statements and their actual meanings and correct version. The apparent mistakes in some conventional terms or statements have also been pointed out.

Conventional terms/ Statements	*Apparent mistakes*	*Actual meaning and correct version*
1. Forces P and Q have a resultant R.	Forces are vectors. Hence they cannot be represented by scalar numbers P, Q and R.	Forces of magnitudes P and Q have a resultant of magnitude R.
2. Two equal forces P, P act at an angle of 60° with each other.	Two equal forces have always the same direction. Hence, they can not be inclined at any non-zero angle with each other.	Two forces each of magnitude P act at an angle of 60° with each other.
3. Interchange of two forces P, Q.	Whenever the line of action of a force, say $\vec{F}$, acting along a line, say OA, is turned through an angle and if the new line of action be OB, then the force acting along OB connot be the same force $\vec{F}$.	Interchange of the magnitudes P and Q of two forces keeping their directions unaltered.
4. Turning a force through an angle θ.	Same as above.	Turning the direction (or line of action) of a force through an angle θ keeping its magnitude unaltered.

		(Obviously, the two forces in two different directions will be different).
5. A force becomes n times as great as before.		The magnitude of a force becomes n times as great as before, its direction remaining unaltered.
6. Two forces are proportional to the areas of two given triangles.	Forces being vector quantities, their ratios can not be defined.	The magnitudes of two forces are proportional to the areas of two given triangles.
7. The moment of a force P about a point at a distance p from the line of action of the force is Pp.	Moment being a vector quantity can not be represented by the scalar quantity Pp.	The magnitude of the moment of a force $\vec{P}$ about a point at a distance p from the line of action of $\vec{P}$ is Pp.

2.9 Solved Examples

Ex. 1. Two equal forces act on a particle. Find the angle between them when the square of their resultant is equal to three times their product. [P.U. 1930]

Let the angle between two equal forces, P, P be α and let R be their resultant.

Hence, $R^2=P^2+P^2+2P\times P\cos\alpha$

$$\Rightarrow 3P\times P=2P^2\,(1+\cos\alpha)$$

$$\Rightarrow\quad \frac{3}{2}=1+\cos\alpha=2\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \cos\frac{\alpha}{2}=\frac{\sqrt{3}}{2}\Rightarrow\frac{\alpha}{2}=30^\circ\Rightarrow\alpha=60^\circ.$$

Hence, the required angle is 60°.

Ex. 2. Forces of magnitudes 2, $\sqrt{3}$, 5, $\sqrt{3}$ and 2 N act respectively at one of the angular points of a regular hexagon towards the five others in order. Find the magnitude and direction of the resultant.

Let the forces of magnitudes 2, $\sqrt{3}$, 5, $\sqrt{3}$ and $2N$ act at A of the regular haxagon $ABCDEF$ along AB, AC, AD, AE and AF respectively.

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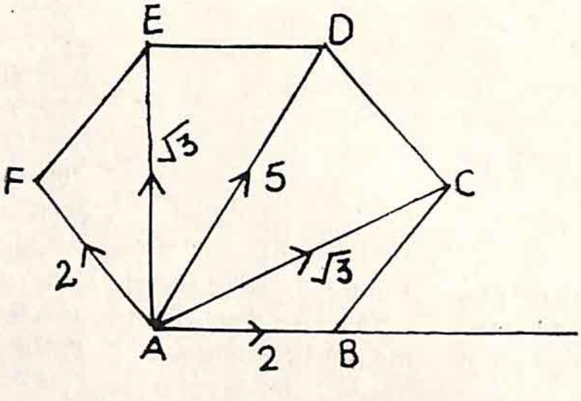


Fig. 12

Clearly, $\angle BAC = \angle CAD = \angle DAE = \angle EAF = 30°$.

Hence, AC, AD, AE and AF make angles 30°, 60°, 90° and 120° with AB. We take the perpendicular lines AB and AE as the X and Y axes respectively and resolve all the forces along these two lines. If X and Y be the sums of the resolved parts of all the forces along AX and AY respectively, then

$$X = 2 + \sqrt{3}\cos 30° + 5\cos 60° + \sqrt{3}\cos 90° + 2\cos 120°$$

$$= 2 + \sqrt{3} \times \frac{\sqrt{3}}{2} + 5 \times \frac{1}{2} + 0 - 2 \times \frac{1}{2} = 5$$

$$Y = \sqrt{3}\sin 30° + 5\sin 60° + \sqrt{3}\sin 90° + 2\sin 120°$$

$$= \sqrt{3} \times \frac{1}{2} + 5 \times \frac{\sqrt{3}}{2} + \sqrt{3} + 2 \times \frac{3}{2} = 5\sqrt{3}.$$

Hence, the resultant R of all the forces is given by

$$R = \sqrt{X^2 + Y^2} = \sqrt{5^2 + (5\sqrt{3})^2} = 10.$$

If θ be the angle made by the resultant with AX, then

$$\tan\theta = \frac{Y}{X} = \frac{5\sqrt{3}}{5} = \sqrt{3} = \tan 60°.$$

$$\therefore \theta = 60°.$$

Hence, the magnitude of the resultant is 10 N and it will act in the direction of the vertex D.

Ex. 3. Four forces represented by $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{CB}$ and $\overrightarrow{CD}$ where $ABCD$ is a quadrilateral act at a point. Show that their resultant is represented by $4\,\overrightarrow{FE}$, where E and F are the mid-points of AC and BD respectively.

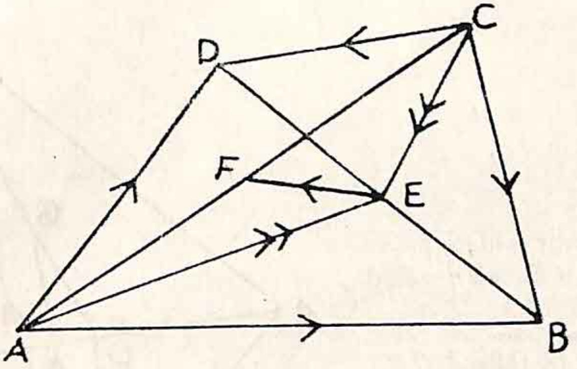


Fig. 13

We have $\overrightarrow{AB}+\overrightarrow{AD}=2\,\overrightarrow{AE}$

and $\overrightarrow{CD}+\overrightarrow{CB}=2\,\overrightarrow{CE}$

Adding we get

$$\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CD}+\overrightarrow{CB}=2\,(\overrightarrow{AE}+\overrightarrow{CE})$$

$$=-2\,(\overrightarrow{EA}+\overrightarrow{EC})$$

$$=-2\times 2\,\overrightarrow{EF}$$

$$=-4\,\overrightarrow{EF}$$

$$=\ 4\,\overrightarrow{FE}.$$

Ex. 4. Prove that the resultant of three forces represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, where A, B, C are any three non-collinear points is $3\,\overrightarrow{OG}$, where G is the centroid of the triangle ABC.

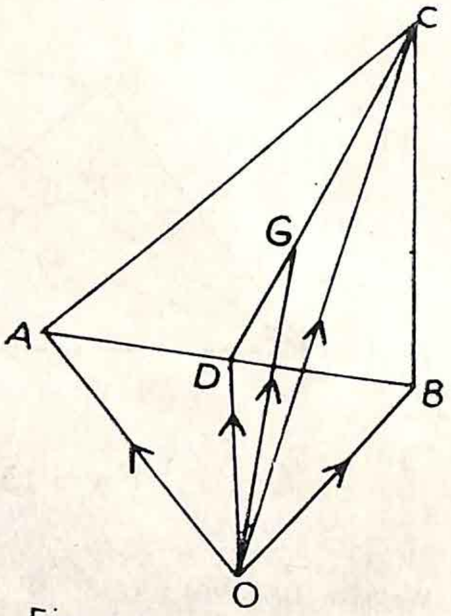


Fig. 14

Let D be the middle point of AB so that CD is a median of $\triangle ABC$.

Now, $\overrightarrow{OA}+\overrightarrow{OB}=2\,\overrightarrow{OD}$.

Also, $2\,\overrightarrow{OD}+\overrightarrow{OC}=(2+1)\,\overrightarrow{OG}=3\,\overrightarrow{OG}$,

since $CG : GD=2 : 1$, G being the centroid of $\triangle ABC$.

Hence, the resultant of three forces $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is represented by $3\,\overrightarrow{OG}$.

Note : The point O in the above problem need not be coplanar with $\triangle\ ABC$.

Ex. 5. The angle of inclination between two forces P and Q is θ. If P and Q be interchanged in position, show that the resultant will be turned through an angle ϕ, where

$$\tan\frac{\phi}{2}=\frac{P-Q}{P+Q}\tan\frac{\theta}{2}.$$ [C.U. 1929]

Let the forces P, Q be represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$ respectively and the resultant R be represented by the diagonal $\overrightarrow{AC}$ of the parallelogram $ABCD$.

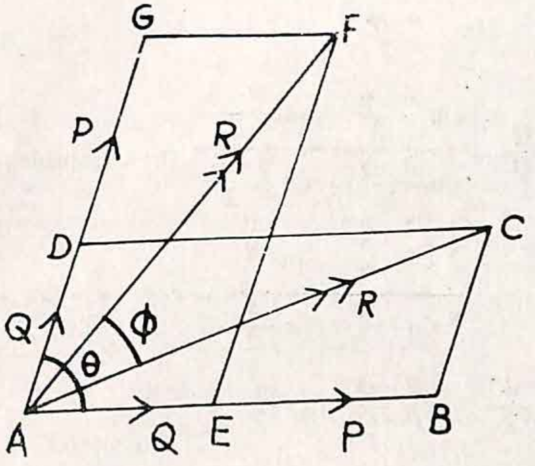


Fig. 15

When the forces are interchanged, let $\overrightarrow{AE}$, $\overrightarrow{AG}$ and $\overrightarrow{AF}$ represent the forces of magnitudes Q, P and R respectively. Then

$\angle GAB=\theta$ and $\angle FAC=\phi$.

Now, $\angle GAF=\angle CAB$, since $\triangle ABC \cong \triangle AFG$.

But $\angle GAF+\angle CAB=\theta-\phi$.

$$\therefore \angle GAF=\angle CAB=\frac{\theta-\phi}{2}.$$

$$\therefore \angle FAB=\phi+\angle CAB=\phi+\frac{\theta-\phi}{2}=\frac{2\phi+\theta-\phi}{2}=\frac{\theta+\phi}{2}.$$

Now, since $AB \parallel GF$,

hence, $\angle GFA=\angle FAB=\dfrac{\theta+\phi}{2}$.

Hence, from $\triangle AGF$, we have

$$\frac{GF}{\sin \angle GAF}=\frac{AG}{\sin \angle GFA}$$

$$\Rightarrow \frac{Q}{\sin\frac{\theta-\phi}{2}}=\frac{P}{\sin\frac{\theta+\phi}{2}}$$

$$\Rightarrow \frac{P}{Q}=\frac{\sin\frac{\theta+\phi}{2}}{\sin\frac{\theta-\phi}{2}}$$

$$\Rightarrow \frac{P-Q}{P+Q}=\frac{\sin\frac{\theta+\phi}{2}-\sin\frac{\theta-\phi}{2}}{\sin\frac{\theta+\phi}{2}+\sin\frac{\theta-\phi}{2}} \text{ (by componendo and dividendo)}$$

$$=\frac{2\cos\frac{\theta}{2}\sin\frac{\phi}{2}}{2\sin\frac{\theta}{2}\cos\frac{\phi}{2}}=\cot\frac{\theta}{2}\tan\frac{\phi}{2}$$

$$\Rightarrow \tan\frac{\phi}{2}=\frac{P-Q}{P+Q}\tan\frac{\theta}{2}.$$

Alternative method of solution :

Since the resultant of the forces P and Q represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$ respectively is R which is represented by $\overrightarrow{AC}$, hence

$$\tan \angle CAB=\frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\Rightarrow \tan\frac{\theta-\phi}{2}=\frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\Rightarrow \frac{\tan\frac{\theta}{2}-\tan\frac{\phi}{2}}{1+\tan\frac{\theta}{2}\tan\frac{\phi}{2}}=\frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\Rightarrow Q\sin\theta\,(1+\tan\frac{\theta}{2}\tan\frac{\phi}{2})=(P+Q\cos\theta)(\tan\frac{\theta}{2}-\tan\frac{\phi}{2})$$

$$\Rightarrow \tan\frac{\phi}{2}\,(Q\sin\theta\tan\frac{\theta}{2}+P+Q\cos\theta)$$

$$=(P+Q\cos\theta)\tan\frac{\theta}{2}-Q\sin\theta$$

$$\Rightarrow \tan\frac{\phi}{2}=\frac{P\tan\dfrac{\theta}{2}+Q\cos\theta\tan\dfrac{\theta}{2}-Q\sin\theta}{P+Q\cos\theta+Q\sin\theta\tan\dfrac{\theta}{2}}$$

$$\Rightarrow \tan\frac{\phi}{2}=\frac{Q\left(\cos\theta\tan\dfrac{\theta}{2}-\sin\theta\right)+P\tan\dfrac{\theta}{2}}{Q\left(\cos\theta+\sin\theta\tan\dfrac{\theta}{2}\right)+P}$$

$$=\frac{\dfrac{Q\left(\cos\theta\sin\dfrac{\theta}{2}-\cos\dfrac{\theta}{2}\sin\theta\right)}{\cos\dfrac{\theta}{2}}+P\tan\dfrac{\theta}{2}}{\dfrac{Q\left(\cos\theta\cos\dfrac{\theta}{2}+\sin\theta\sin\dfrac{\theta}{2}\right)}{\cos\dfrac{\theta}{2}}+P}$$

$$=\frac{-Q\tan\dfrac{\theta}{2}+P\tan\dfrac{\theta}{2}}{Q+P}$$

$$=\frac{P-Q}{P+Q}\tan\frac{\theta}{2}.$$

Ex. 6. Forces act through the angular points of a triangle perpendicular to the opposite sides and are proportional to the cosines of the corresponding angles. Show that their resultant is proportional to

$$\sqrt{1-8\cos A\cos B\cos C}.$$

Let the forces P, Q and S act at A,B,C respectively perpendicular to the sides BC, CA and AB respectively.

Let R be the resultant of these forces and let R make an angle θ with BC. Let D, E and F be the points of intersection of the lines of action of the forces P, Q and S with the sides BC, CA and AB respectively. We choose the line BC as the X-axis and a line perpendicular to it through B as the Y-axis.

Resolving all the forces along X and Y axes, we get,

$$R\cos\theta=Q\cos\angle EBC-S\cos\angle FCB$$

$$=Q\cos(90^\circ-C)-S\cos(90^\circ-B)$$

$$=Q\sin C-S\sin B. \qquad \ldots(1)$$

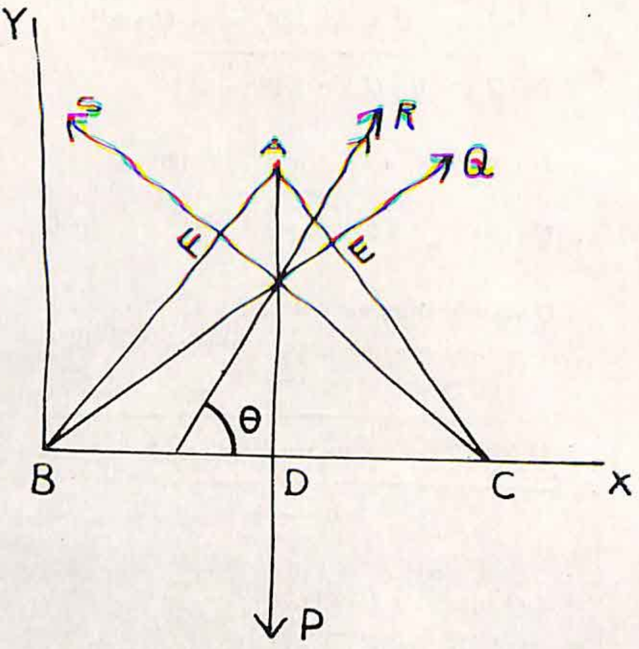


Fig. 16

$$R \sin \theta = Q \sin \angle EBC + S \sin \angle FCB - P$$
$$= Q \sin (90^\circ - C) + S \sin (90^\circ - B) - P$$
$$= Q \cos C + S \cos B - P. \qquad \ldots(2)$$

Squaring (1) and (2) and adding, we get

$$R^2 = (Q \sin C - S \sin B)^2 + (Q \cos C + S \cos B - P)^2$$
$$= Q^2 (\cos^2 C + \sin^2 C) + P^2 + S^2 (\sin^2 B + \cos^2 B)$$
$$+ 2\, SQ (\cos B \cos C - \sin B \sin C)$$
$$- 2\, PQ \cos C - 2\, PS \cos B$$
$$= P^2 + Q^2 + S^2 + 2\, SQ \cos (B + C) - 2\, PQ \cos C - 2\, SP \cos B$$
$$= P^2 + Q^2 + S^2 - 2\, (SQ \cos A + PQ \cos C + SP \cos B).$$

Now, it is given that

$$\frac{P}{\cos A} = \frac{Q}{\cos B} = \frac{S}{\cos C} = K \text{ say.}$$

$$P=K\cos A,\ Q=K\cos B,\ S=K\cos C$$

$$\therefore\ R^2=K^2[\cos^2A+\cos^2B+\cos^2C-6\cos A\cos B\cos C]$$

$$=K^2\left[\frac{1}{2}(2+\cos 2A+\cos 2B)+\cos^2C-6\cos A\cos B\cos C\right]$$

$$=K^2\,[1+\cos(A+B)\cos(A-B)+\cos^2C-6\cos A\cos B\cos C]$$

$$=K^2\,[1+\cos C\,\{\cos(A-B)+\cos(A+B)\}-6\cos A\cos B\cos C]$$

$$=K^2\,[1-2\cos A\cos B\cos C-6\cos A\cos B\cos C]$$

$$=K^2\,[1-8\cos A\cos B\cos C]$$

Hence, $R=K\sqrt{1-8\cos A\cos B\cos C}$

i.e. the resultant is proportional to $\sqrt{1-8\cos A\cos B\cos C}$.

Ex. 7. If P be the ortho-centre and O, the circumcentre of the triangle ABC, prove that (i) $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}=2\,\overrightarrow{PO}$

(ii) $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OP}$

Let G be the centroid of $\triangle ABC$. It is known from Geometry that O, G, P are collinear. Also $PG=2\ GO$ i.e. $OP=3\,OG$. Now, from Ex. 4, we have

$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}=3\,\overrightarrow{PG}=6\,\overrightarrow{GO}=2\,\overrightarrow{PO}$ and

$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=3\,\overrightarrow{OG}=\overrightarrow{OP}$

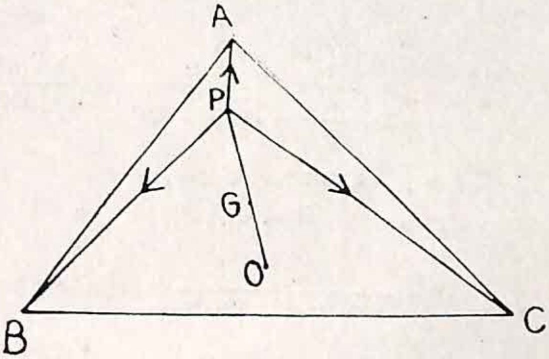


Fig 17

Ex. 8. A, B, C are points on the circumference of a circle. Forces act along AB and BC being inversely proportional to these line segments in magnitude. Show that their resultant acts along the tangent at B.

[D.U. Hons. '64]

Let the forces of magnitudes P and Q act along AB and BC respectively and let R be their resultant acting at B making an angle θ with BC. We denote the sides opposite to the angles A, B and C by a, b and c respectively. It is given that $P=\frac{K}{c}$ and $Q=\frac{K}{a}$, where K is a constant.

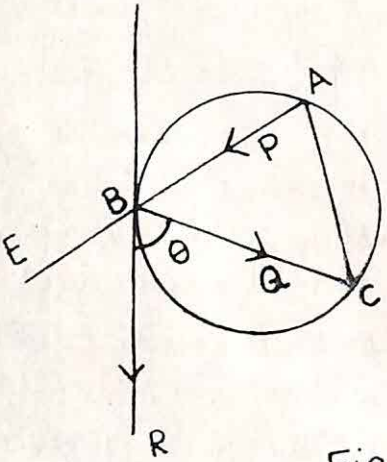


Fig. 18

Since the resultant R makes and angle θ with BC, hence

$$\tan\theta=\frac{P\sin\angle EBC}{Q+P\cos\angle EBC}$$

$$=\frac{\frac{K}{c}\sin(180°-B)}{\frac{K}{a}+\frac{K}{c}\cos(180°-B)}=\frac{a\sin B}{c-a\cos B}$$

But $a\cos B+b\cos A=c$

$\therefore\ c-a\cos B=b\cos A$

Also $\frac{a}{\sin A}=\frac{b}{\sin B}\Rightarrow a\sin B=b\sin A.$

Hence, $\tan\theta=\frac{b\sin A}{b\cos A}=\tan A.$

$\therefore\ \theta=A$.

But the tangent at B makes an angle A with the chord BC.

Hence, the resultant R acts along the tangent to the circle at B.

Ex. 9. The tangent and normal at any point P of a parabola meet the axis of the parabola in T and N respectively. Prove that the resultant of forces $\overrightarrow{PT}$ and $\overrightarrow{PN}$ passes through the focus of the parabola.

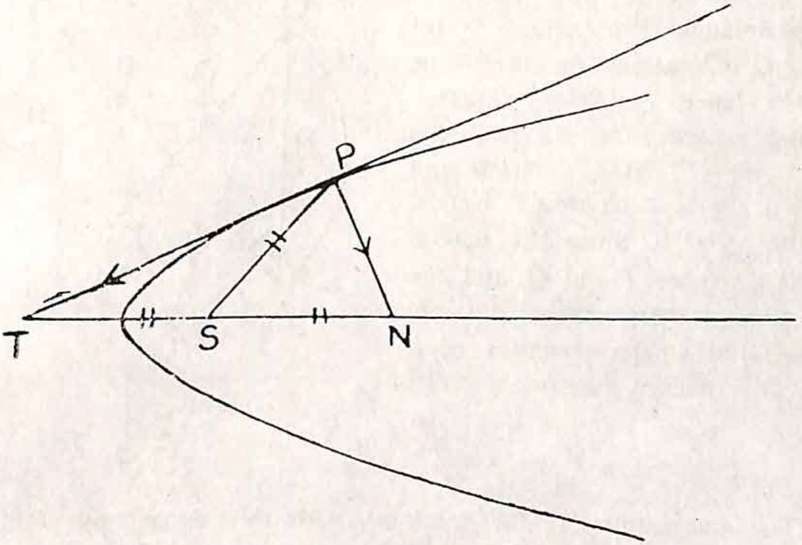


FIG. 19

Let S be the focus of the parabola. From the properties of the tangent and normal to a parabola, we have

$\angle SPT = \angle PTS$ and $\angle SPN = \angle SNP$

$\therefore\ TS = SP = SN$

i.e. S is the middle point of TN.

Hence, $\overrightarrow{PT} + \overrightarrow{PN} = 2\,\overrightarrow{PS}$

i.e. the resultant of PT and PN passes through S, the focus.

Ex. 10. Two coplanar non-parallel forces act at two given points of a body. If they are turned through the same angle in the same sense about their respective points of application, prove that the resultant is constant in magnitude and passes through a fixed point.

Let the two forces P and Q act at A and B and meet at O. We draw a circle through O, A and B. Let the resultant of P and Q acting along OA and OB respectively pass through C, a point on the circle. Let the force P be turned through an angle α in the anticlockwise direction and let $O'A$ be the new position of the force P. Then $\angle O'AO = \alpha$. We join $O'B$.

Multiplying (1) by n^2 and subtracting from (2), we get

$$R^2\,(1-n^2) = 2n^2\,PQ\left(\cos\frac{\theta}{2} - \cos\theta\right) \qquad ...(4)$$

Multiplying (2) by n^2 and subtracting from (3), we get

$$R^2\,(1-n^2) = 2n^4\,PQ\left(\cos\frac{\theta}{4} - \cos\frac{\theta}{2}\right) \qquad ...(5)$$

From (4) and (5), we get

$$n^2\left(\cos\frac{\theta}{4} - \cos\frac{\theta}{2}\right) = \cos\frac{\theta}{2} - \cos\theta$$

$$\Rightarrow n^2 = \frac{\cos\theta - \cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \cos\frac{\theta}{4}} = \frac{-2\sin\frac{3\theta}{4}\sin\frac{\theta}{4}}{-2\sin\frac{3\theta}{8}\sin\frac{\theta}{8}}$$

$$= \frac{4\sin\frac{3\theta}{8}\cos\frac{3\theta}{8}\sin\frac{\theta}{8}\cos\frac{\theta}{8}}{\sin\frac{3\theta}{8}\sin\frac{\theta}{8}}$$

$$= 4\cos\frac{3\theta}{8}\cos\frac{\theta}{8} = 2\left(\cos\frac{\theta}{2} + \cos\frac{\theta}{4}\right)$$

$$= 2\left(\cos\frac{\theta}{4} + 2\cos^2\frac{\theta}{4} - 1\right)$$

$$\Rightarrow 4\cos^2\frac{\theta}{4} + 2\cos\frac{\theta}{4} - 2 - n^2 = 0$$

$$\Rightarrow \cos\frac{\theta}{4} = \frac{-2 \pm \sqrt{4 + 16\,(2+n^2)}}{8}$$

Rejecting the negative value of $\cos\frac{\theta}{4}$, which is not physically possible, we get

$$\cos\frac{\theta}{4} = \frac{-1 + \sqrt{9 + 4n^2}}{4}$$

$$\Rightarrow \theta = 4\cos^{-1}\left(\frac{\sqrt{9+4n^2} - 1}{4}\right).$$

Ex. 12. A quadrilateral *ABCD* is insctibed in a circle of centre 0. Forces proportional to the areas of the traingles *BCD*, *CDA*, *DAB* and *ABC* act respectively along *AO*, *OB*, *CO* and *OD*. Show that the forces are in equilibrium.

Let α, β, γ and δ be the areas of $\triangle\ BCD$, $\triangle\ CDA$, $\triangle\ DAB$ and $\triangle\ ABC$ respectively. Then the forces of magnitudes $K\alpha$. $K\beta$, $K\gamma$ and $K\delta$, where K is some non-zero constant, act along AO, OB, CO and OD respectively. Let r be the radius of the circle and let AC and BD meet at L. The unit vectors along AO, OB, CO and OD can be written as

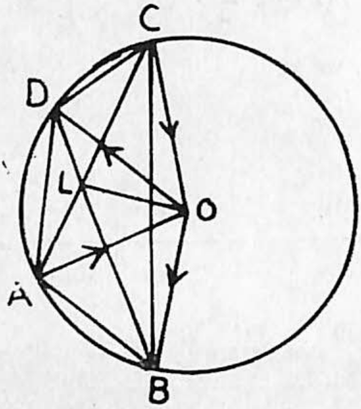


Fig. 21

$\frac{\overrightarrow{AO}}{r}$, $\frac{\overrightarrow{OB}}{r}$, $\frac{\overrightarrow{CO}}{r}$ and $\frac{\overrightarrow{OD}}{r}$ respectively. Hence, the forces along AO, OB, CO and OD can be written as $\frac{K\alpha}{r}\overrightarrow{AO}$, $\frac{K\beta}{r}\overrightarrow{OB}$, $\frac{K\gamma}{r}\overrightarrow{CO}$ and $\frac{K\delta}{r}\ \overrightarrow{OD}$ respectively. Now, we have

$$\frac{\text{Area of } \triangle\ ACD}{\text{Area of } \triangle\ AOB} = \frac{h_1}{h_2},$$

where h_1 is the height of D from AC and h_2 is the height of B from AC.

$$\text{Hence, } \frac{\beta}{\delta} - \frac{h_1}{h_2} = \frac{DL}{BL}$$

Similarly,

$$\frac{\text{area of } \triangle\ BCD}{\text{area of } \triangle\ DAB} = \frac{CL}{AL}.$$

$$\rightarrow \frac{\alpha}{\gamma} = \frac{CL}{AL}.$$

Since $\dfrac{DL}{BL} = \dfrac{\beta}{\delta} = \dfrac{\frac{K\beta}{r}}{\frac{K\delta}{r}}$,

hence, $\dfrac{K\beta}{r}\overrightarrow{OB} + \dfrac{K\delta}{r}\overrightarrow{OD} = \dfrac{K}{r}(\beta+\delta)\,\overrightarrow{OL}$ ...(1)

and since $\dfrac{CL}{AL} = \dfrac{\alpha}{\gamma} = \dfrac{\frac{K\alpha}{r}}{\frac{K\gamma}{r}}$,

hence, $\dfrac{K\alpha}{r}\overrightarrow{AO} + \dfrac{K\gamma}{r}\overrightarrow{CO} = \dfrac{K}{r}(\alpha+\gamma)\,\overrightarrow{LO}$ ...(2)

Adding (1) and (2), we get

$$\frac{K\alpha}{r}\overrightarrow{AO} + \frac{K\gamma}{r}\overrightarrow{CO} + \frac{K\beta}{r}\overrightarrow{OB} + \frac{K\delta}{r}\overrightarrow{OD} = \frac{K}{r}\overrightarrow{OL}\,(\beta+\delta-\alpha-\gamma)$$

But $\alpha + \gamma =$ area of $\triangle\ BCD$ + area of $\triangle\ DAB$ = area of quadrilateral $ABCD$,

and $\beta + \delta =$ area of $\triangle\ CDA$ + area of $\triangle\ ABC$ = area of quadrilateral $ABCD$.

Hence, $\beta + \delta - \alpha - \gamma = 0$.

∴ The forces acting along AO, OB, CO and OD are in equilibrium,

Ex. 13. $ABCDE$ is a regular pentagon. Forces of magnitudes $n^2 + 2n$, n, n and $2n + 5$ Newtons act along AB, AC, AD and AE respectively. Show that their resultant is

$\sqrt{n^4 + 5n^3 + 15n^2 + 25n + 25}$ Newton.

From the figure, we have

$\angle ABC = \angle BCD = \angle CDE = \angle DEA = \angle EAB = 108^\circ$

$\therefore \angle BAC = \angle BCA = \frac{1}{2}(180^\circ - 108^\circ) = 36^\circ$

$\angle DEA = 108^\circ \therefore \angle EAD = \angle EDA = \frac{1}{2}(180^\circ - 108^\circ) = 36^\circ$

$\therefore \angle DAC = 108^\circ - 72^\circ = 36^\circ$.

Let R be the resultant of all the forces and let R make an angle θ with AB.

Resolving all the forces along AB and perpendiculat to AB, we get

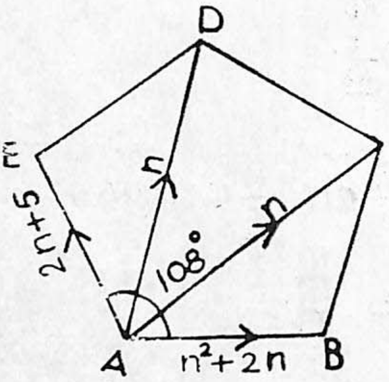


Fig. 22

$$R \cos \theta = (n^2 + 2n) + n \cos 36^\circ + n \cos 72^\circ + (2n + 5) \cos 108^\circ$$
$$= (n^2 + 2n) + n \cos 36^\circ + n \cos 72^\circ - (2n + 5) \cos 72^\circ$$

and

$$R \sin \theta = n \sin 36^\circ + n \sin 72^\circ + (2n + 5) \sin 108^\circ$$
$$= n \sin 36^\circ + n \sin 72^\circ + (2n + 5) \sin 72^\circ.$$

Hence,

$$R^2 = (n^2 + 2n)^2 + n^2 (\cos^2 36^\circ + \sin^2 36^\circ)$$
$$+ n^2 (\cos^2 72^\circ + \sin^2 72^\circ)$$
$$+ (2n + 5)^2 (\cos^2 72^\circ + \sin^2 72^\circ) + 2n (n^2 + 2n) \cos 36^\circ$$
$$+ 2n (n^2 + 2n) \cos 72^\circ - 2 (2n + 5) (n^2 + 2n) \cos 72^\circ$$
$$+ 2n^2 \cos 36^\circ \cos 72^\circ - 2n (2n + 5) \cos 36^\circ \cos 72^\circ$$
$$- 2n (2n + 5) \cos^2 72^\circ + 2n^2 \sin 36^\circ \sin 72^\circ$$
$$+ 2n (2n + 5) \sin 36^\circ \sin 72^\circ + 2n (2n + 5) \sin^2 72^\circ$$
$$= (n^2 + 2n)^2 + n^2 + n^2 + (2n + 5)^2 + 2n (n^2 + 2n) \cos 36^\circ$$
$$+ 2n (n^2 + 2n) \cos 72^\circ - 2 (2n + 5) \times (n^2 + 2n) \cos 72^\circ$$
$$+ 2n^2 \cos (72^\circ - 36^\circ) - 2n (2n + 5) \cos (36^\circ + 72^\circ)$$
$$- 2n (2n + 5) \cos (2 \times 72^\circ)$$
$$= n^4 + 4n^3 + 10n^2 + 20n + 25 + 2n (n^2 + 2n) \cos 36^\circ$$
$$+ 2n (n^2 + 2n) \cos 72^\circ - 2 (2n + 5) (n^2 + 2n) \cos 72^\circ$$
$$+ 2n^2 \cos 36^\circ + 2n (2n + 5) \cos 72^\circ + 2n (2n + 5) \cos 36^\circ$$
$$= n^4 + 4n^3 + 10n^2 + 20\,n + 25$$
$$+ \cos 36^\circ (2n^3 + 4n^2 + 2n^2 + 4n^2 + 10n)$$

$$+ \cos 72^\circ (2n^3 + 4n^2 - 4n^3 - 8n^2 - 10n^2 - 20n + 4n^2 + 10n)$$
$$= n^4 + 4n^3 + 10n^2 + 20n + 25 + 2n + \cos 36^\circ (2n^3 + 10n^2 + 10n) - \cos 72^\circ (2n^3 + 10n^2 + 10n)$$
$$= n^4 + 4n^3 + 10n^2 + 20n + 25 + 2n (n^2 + 5n + 5) \times (\cos 36^\circ - \cos 72^\circ)$$
$$= n^4 + 4n^3 + 10n^2 + 20n + 25 + 2n (n^2 + 5n + 5) \times \left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right)$$
$$= n^4 + 4n^3 + 10n^2 + 20n + 25 + n^3 + 5n^2 + 5n$$
$$= n^4 + 5n^3 + 15n^2 + 25n + 25$$

Hence, $R = \sqrt{n^4 + 5n^3 + 15n^2 + 25n + 25}$.

Ex. 14. Two equal weights of 5 N are attached to the ends of a thin string which passes over three smooth pegs in a wall arranged in the from of an equilateral triangle with one side horizontal. Find the pressure on each peg.

Let A, B, C be the position of the pegs. The tension of the string will be 5 N, since the pegs are smooth.

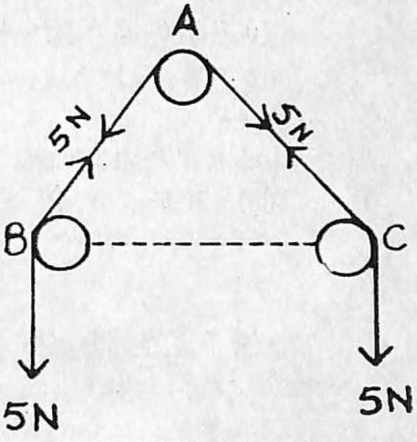


Fig. 23

The pressure at B is the resultant of two forees $5N$ and $5N$ inclined at an angle of 150°. Similarly the pressure at C will be the same as that at B. The pressure at A will be the resultant of two forces 5 N and 5 N inclined at an angle of 60°. Let R be the pressure at B or C and S be the pressure at A.

Hence, $R^2 = 5^2 + 5^2 + 2 \times 5 \times 5 \cos 150^\circ$

$$= 25 + 25 + 50 \times \left(-\frac{\sqrt{3}}{2}\right)$$

$= 50\left(1-\frac{\sqrt{3}}{2}\right)= 25\ (2-\sqrt{3}).\ \Big)$

$\therefore R = 5\sqrt{2-\sqrt{3}}$

and $S^2=5^2+5^2+2\times5\times5\cos 60°$

$=75$

$\therefore S=5\sqrt{3}.$

Hence, the pressures on pegs A,B,C are $5\sqrt{2-\sqrt{3}}\,N$, $5\sqrt{2-\sqrt{3}}\,N$ and $5\sqrt{3}\,N$ respectively. Since the component forces acting at each peg are equal, hence the resultant force at each peg will bisect the angle between the two parts of the string on each side of the peg.

EXERCISE 2.1

1. In the following problems, R is the resultant of two forces P and Q inclined at an angle α with each other. Find

(i) R if $P=3\ N$, $Q=4\ N$ and $\alpha=90°$ [Ans. $5N$]

(ii) P if $R=15\ N$, $Q=10\ N$ and $\alpha=60°$

[Ans. $5(\sqrt{6}-1)N$]

(iii) R if $P=12\ N$, $Q=12\ N$ and $\alpha=120°$ [Ans. $12\ N$]

(iv) α if $P=7\ N$, $Q=3\ N$ and $R=5\ N$

$\left[\text{Ans. } \cos^{-1}\left(-\frac{11}{14}\right)\right]$

(v) R if $P=2$N, $Q=1N$ and $\alpha=\tan^{-1}\frac{3}{4}$

$\left[\text{Ans. } \sqrt{\frac{41}{5}}\ \text{N}\right]$

2. Show that the resultant of two forces each equal to P, and inclined at an angle of 120° is also equal to P.

3. Find the angle between two equal forces P, P when the square of their resultant is equal to $(2-\sqrt{3})$ times their product.

[Ans. 150°]

4. With two forces acting at a point, the maximum effect is obtained when their resultant is 4 N. If they act at right angles, their resultant is 3 N. Find the forces.

[Ans. $(2\pm\frac{1}{2}\sqrt{2})\ N$]

5. The resultant of two forces P and Q is equal to P in magnitude, and that of two forces $2P$ and Q (acting in the same direction as before) is also equal to P. Find the magnitude of Q and prove that the direction of Q makes an angle of 150° with that of P.

[Ans. $\sqrt{3}\,P$]

6. Two forces act at a point and are such that if the direction of one is reversed, the direction of the resultant is turned through a right angle. Prove that the two forces must be equal in magnitude.

7. Two forces P and Q act at such an angle that the resultant R is equal to P. Show that if P is doubled, the new resultant is at right angles to the force Q.

8. The resultant of two forces P and Q is of magnitude R. Show that if P is doubled, Q remaining unaltered, then the new resultant S will be of magnitude
$\sqrt{2P^2+2R^2-Q^2}$.

9. If the resultant of two forces acting on a particle be at right angles to one of them, and its magnitude be half the magnitude of the other, show that the ratio of the larger force to the smaller is $2 : \sqrt{3}$.

10. The resultant of two forces P and $2P$ acting at a point is perpendicular to P. Find the angle between the forces.

[Ans. 120°]

11. Find the angle between two equal forces when their resultant is the third equal force P. [*P.U.* 1930]

[Ans. 120°]

12. Two forces of magnitudes $3P$ and $2P$ respectively have a resultant R. If the first force is doubled, the magnitude of the resultant is also doubled. Find the angle between the forces.

[*C.U.* 1932]

[Ans. 120°]

13. At what angle must two forces of 10 N and 10 N be inclined if they are balanced by a force of 5 N?

$$\left[\text{Ans. } \cos^{-1}\left(-\frac{7}{8}\right)\right]$$

14. Two forces acting on a particle are not on a line. One is halved and the other is decreased by 10 N. Find the latter force if the resultant remains unaltered in direction.

[Ans. 20 N]

15. The resultant of two forces P and Q is R. If Q is doubled, the new resultant is prependicular to P. Prove that $Q=R$.

16. The resultant of two forces P and Q $(P>Q)$ acting at right angles is R. If P and Q be each descreased by 3 N, R is decreased by 4 N, and is now equal to the difference of the original values of P and Q. Find P and Q.

[Ans. $P=4$ N, $Q=3$ N]

17. If the resultant of two equal forces inclined at an angle 2θ is twice as great as when they are inclined at an angle 2ϕ, prove that $\cos\theta=2\cos\phi$.

18. Two forces P and Q have a resultant R. If P is increased, the new resultant bisects the angle between R and P. Find the increase of P.

[Ans. R]

19. Two forces $P+Q$ and $P-Q$ make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that $P\tan\theta=Q\tan\alpha$. [*P.U.* 1931]

20. Two forces P and Q inclined at an angle of 120° have a resultant R. When they are inclined at an angle of 60°, the resultant becomes n times as great as before. Prove that

$$P=\frac{R}{2\sqrt{2}}\left(\sqrt{3n^2-1}+\sqrt{3-n^2}\right)$$

and $$Q=\frac{R}{2\sqrt{2}}\left(\sqrt{3n^2-1}-\sqrt{3-n^2}\right).$$

21. Forces P and Q whose resultant is R act at a point O. If any transversal cut the lines of action of the forces P, Q and R at the points L,M,N respectively, show that

$$\frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}.$$ [C.U. '47; B.H.U. '44]

22. Two forces P and Q acting on a particie at an angle α have a resultant $(2K+1)\sqrt{P^2+Q^2}$. When they act at an angle $90^\circ-\alpha$, the resultant becomes $(2K-1)\sqrt{P^2+Q^2}$.
Prove that

$$\tan\alpha=\frac{K-1}{K+1}.$$ [B.H.U. '46]

23. The resultant of two forces P and Q is $\sqrt{3}\,Q$ and makes an angle 30° with the direction of P. Show that either $P=Q$ or $P=2Q$.

24. If the resultant of the forces P and Q be equal to that of the forces $P+S$ and Q—S acting at the same angle ($S \neq Q-P$), show that the magnitude of the resultant is $P+Q$.

25. If the resultant R of two forces P and Q inclined to one another at any given angle make an angle θ with the direction of P, show that the resultant of the forces $P+R$ and Q acting at the same angle will make an angle $\frac{1}{2}\theta$ with the direction of $P+R$.

[*B.U.*'26,'29; *B.E.*'32]

26. Show that the system of forces represented by the lines joining any point to the angular points of a triangle is equivalent to the system represented by straight lines drawn from the same point to the middle points of the sides of the triangle.

27. The resultant of two forces P and Q acting at a certain angle is X and that of P and R acting at the same angle is also X. The resultant of Q and R again acting at the same angle is Y. Prove that

$$P=\sqrt{X^2+QR}=\frac{QR\,(Q+R)}{Q^2+R^2-Y^2},$$

$P+Q+R=0$ and $Y=X$.

28. Two forces P and Q act at an angle α and have a resultant R. If each force is increased by R, prove that the new resultant makes with R an angle whose tangent is

$$\frac{(P-Q)\sin\alpha}{P+Q+R+(P+Q)\cos\alpha}.$$

[*P.U.*' 43; *B.H.U.*' 43]

29. The resultant of two forces is the same when their directions are inclined at an angle ϕ as when they are inclinded at an angle

$\frac{\pi}{4}-\phi$. Prove that $\tan\phi=\sqrt{2}-1$.

30. Two forces P and Q acting respectively along two different straight lines OA and OB have a resultant perpendicular to OA. If two forces P' and Q' acting respectively along the same two straight lines have a resultant perpendicular to OB, show that $PP'=QQ'$.

31. The resultant of the forces P and Q is R. If Q is doubled, R is doubled and if Q is reversed, R is again doubled. Show that
$P:Q:R=\sqrt{2}:\sqrt{3}:\sqrt{2}$. [Bombay '34]

32. Two forces P and Q acting respectively along the straight lines OA and OB which are inclined at an angle α to one another $(\alpha \neq \pi)$ have a resultant R making an angle θ with OA. If Q be changed to Q', the resultant changes to R' making an angle θ' with OA. Show that

$$\frac{R'}{R} = \frac{\sin(\alpha-\theta)}{\sin(\alpha-\theta')}.$$

33. Two forces given in magnitude act each through a fixed point and are inclined at a constant angle θ with each other. Prove that their resultant passes through a fixed point.

34. Two forces P, Q act at a point along two straight lines making an angle α with each other and R is their resultant. Two other forces P', Q' acting along the same two lines have a resultant R'. Prove that the angle between the lines of action of the resultants is

$$\cos^{-1}\left(\frac{P\,P'+Q\,Q'+(P\,Q'+P'\,Q)\cos\alpha}{R\,R'}\right).$$

35. The resultant of two forces P and Q is R, nR and $(n+2)\,R$ according as the angles between the two forces are $90°$, θ and $90°-\theta$ $(\theta \neq 45°)$ respectively. Prove that

$$n = \frac{3\cos\theta+\sin\theta}{\sin\theta-\cos\theta}.$$

36. Two forces R and S act at a point along two straight lines inclined at an angle θ and F is their resultant. Two other forces R' and S' acting along the same lines have a resultant F'. If ϕ be the angle between the lines of action of F and F', prove that

$$\sin^2\phi = \frac{(R^2\,S'^2-2\,RR'\,SS'+R'^2\,S^2)\sin^2\theta}{F^2\,F'^2}.$$

[*C.U.* '46].

37. Show that the resultant of two forces of magnitudes sec B and sec C acting along AB and AC respectively of any triangle ABC is a force of magnitude tan A+tan C along AD, where D is the foot of the perpendicular from A on BC.

38. Two forces, P and Q, acting along intersecting lines have a resultant R. If the force Q is reversed in direction and changed into $\dfrac{R^2-P^2}{Q}$, show that the resultant is still of magnitude R.

39. Two forces acting at a point are represented in magnitude and direction by $\overrightarrow{A_1B_1}$ and $\overrightarrow{A_2B_2}$. Show that their resultant is $2\overrightarrow{AB}$, where A and B are the middle points of A_1A_2 and B_1B_2 respectively.

40. P is a point in the plane of the triangle ABC, and I is the in-centre. Show that the resultant of the forces represented by $PA \sin A$, $PB \sin B$ and $PC \sin C$ along PA, PB and PC respectively is

$$4\,\text{PI} \cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2} \text{ along P I.}$$

41. ABC is an equilateral triangle and G is the centroid of the triangle. Forces of 8, 8 and 16 N act at G along GB, GC and GA respectively. Find the magnitude and direction of their resultant.

[Ans. 8 N along GA].

42. Show that the greater the angle between the lines of action of two forces acting at a point, the less will be their resultant.

43. If 4 N be the greatest value of the resultant of two forces and 2 N its least value, find their resultant when the forces act at right angles to each other.

[Ans. $\sqrt{10}\ N$]

44. If two forces P, Q $(P>Q)$ be inclined to each other at an angle 150°, find the ratio between them when the resultant equals the less.

[Ans. $\frac{P}{Q}=\sqrt{3}$]

45. Two non-zero forces acting at a point have got their resultant $2\sqrt{2}\ N$ when acting at right angles, and $\sqrt{6}\ N$ when acting at an angle of 120°. Find the magnitudes of their greatest and least resultants.

[Ans. Greatest resultant$=2\sqrt{3}\ N$ and
least resultant$=2\ N$]

46. Two equal forces act at a point, first at an angle of $180°-A$ and afterwards at an angle of $180°-2A$. Show that the ratio of the resultant in the latter case to that in the former is equal to $2 \cos \frac{1}{2} A$.

47. Three forces P, Q, R in one plane act on a particle. The angles between R and Q, P and R, and P and Q are α, β and γ respectively. Show that their resultant is

$$\sqrt{P^2+Q^2+R^2+2\,QR\cos\alpha+2\,RP\cos\beta+2\,PQ\cos\gamma}\ .$$

[Delhi '31]

48. If the forces of magnitudes P, Q and R act at a point parallel to and in the directions of the sides BC, CA and AB of a triangle ABC respectively, prove that the magnitude of their resultant is

$$\sqrt{P^2+Q^2+R^2-2\,(QR\cos A+RP\cos B+PQ\cos C)}\ .$$

[Madras '36]

49. Prove that any force in the plane of a triangle ABC can be resolved into three components acting along the sides of the triangle.

In particular, if E and F are the feet of the perpendiculars from B and C upon the opposite sides of the triangle ABC, show that a force P acting along EF can be replaced by $P \cos A$, $P \cos B$ and $P \cos C$ acting along the sides of the triangle.

[*B.E.* ' 35].

50. The sides BC, CA and AB of a triangle ABC are 2, $2\sqrt{2}$ and $(\sqrt{6}+\sqrt{2})$ cm respectively. Find the magnitude and the inclination to BC of the resultant of the following forces acting at a point:

1 N in the direction BC, 2 N in the direction CA and 3 N in the direction BA.

$$\left[\text{Ans. } \frac{\sqrt{23+6\sqrt{3}}}{\sqrt{2}},\ \tan^{-1}\left(\frac{4+\sqrt{3}}{2+\sqrt{2}+\sqrt{3}}\right)\right]$$

51. Forces of magnitudes p, q and r N act at a point in the directions parallel to and in the direction of the sides BC, BA and CA respectively of an equilateral triangle. Show that

$$s^2+pr=p^2+q^2+r^2+pq+qr \text{ and } \tan\theta=\frac{\sqrt{3}\,(q+r)}{2p+q-r},$$

where s is the resultant of p, q and r and θ is the inclination of the resultant with $B\,C$.

52. Forces proportional to $\cos A$, $\cos B$ and $\cos C$ act at a point in directions parallel to the sides $B\,C$, CA and $A\,B$ respectively of a triangle ABC. Show that the magnitude of the resultant is proportional to PO, where P is the ortho-centre and O is the circum-centre of the triangle.

53. Two forces act along the sides CA and CB of a triangle ABC, their magnitudes being proportional to $\cos A$ and $\cos B$. Prove that their resultant is proportional to $\sin C$, and its direction divides the angle C into two portions $\frac{1}{2}(C+B-A)$ and $\frac{1}{2}(C+A-B)$.

[Punjab ' 24]

54. Three forces P, Q and R meet at a point and the resultant of P and Q is $7N$ acting at an angle $\cos^{-1}\frac{11}{14}$ with P. The resultant of P and R is also $7N$ at an angle $\cos^{-1}\left(-\frac{1}{7}\right)$ with P and that of Q and R is

$\sqrt{129}N$ at an angle $\tan^{-1}\left(-\frac{13}{\sqrt{3}}\right)$ with P. Find P, Q, R in magnitude and direction, it being given that Q and R are on the same side of the line of action of P.

[Ans. $P=3N$, $Q=5N$ at an angle of 60° with P,
$R=8N$ at an angle of 120° with P]

55. If O be the circum-centre of the triangle ABC and if forces act along OA, OB and OC respectively proportional to BC, CA and AB, show that their resultant passes through the in-centre.

56. D,E,F are respectively the mid-points of the sides BC, CA and AB of a triangle ABC and O is any point in its plane. Show that

(i) $\overrightarrow{DA}+\overrightarrow{EB}+\overrightarrow{FC}=0$ [Delhi '37; B.H.U. '40]

(ii) $\overrightarrow{OE}+\overrightarrow{OF}+\overrightarrow{DO}=\overrightarrow{OA}$

(iii) $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OF}$

(iv) $\overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}+\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OF}=0$.

57. A,B,C are three fixed points and P is a point such that the resultant of forces $\overrightarrow{PA}$ and $\overrightarrow{PB}$ always passes through C. Show that the locus of P is a straight line.

58. The greatest resultant which two forces can have is P and the least is Q. Show that if they act at an angle θ, the resultant is

$$\left(P^2\cos^2\frac{\theta}{2}+Q^2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}.$$

59. If one of the two forces acting on a particle be double that of the other, and if θ be the angle between the directions of the resultant and the greater force, show that $\theta \ngtr \frac{\pi}{6}$.

60. If the greatest possible resultant of two forces P and Q is n times the least, show that the angle α between P and Q when their resultant is twice the square root of their product is given by

$$\tan^2\frac{\alpha}{2}=\frac{3-n^2}{3n^2-5}.$$

61. Forces of 2,1,4,5,3 and 1 N act at a point A in directions AB, AC, AD, AE, AF and AG where $\angle BAC=30°$, $\angle CAD=30°$, $\angle DAE=90°$, $\angle EAF=30°$ and $\angle FAG=120°$.

Find the magnitude of their resultant and the inclination of its direction to AB.

$$\left[\text{Ans. } \sqrt{30+3\sqrt{3}}\,N,\ \tan^{-1}\left(-\ \frac{18+11\sqrt{3}}{13}\right)\right]$$

62. Six horizontal wires are attached to a telephone post and exert the following tensions on it:

$4N$ towards east, $5N$ towards north-east, $3N$ towards 30° west of north, $10N$ towards west, $3N$ towards south-west and $12N$ towards 60° south of east. Show that the magnitude of the resultant is $\sqrt{67-3\sqrt{2}-9\sqrt{6}}\,N$.

63. Three forces represented by $\overrightarrow{PA}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$ diverge from the point P, and three forces represented by $\overrightarrow{AQ}$, $\overrightarrow{BQ}$ and $\overrightarrow{CQ}$ converge to the point Q. Show that the resultant of the six forces is represented by $3\overrightarrow{PQ}$.

64. Two equal weights of P Newton each are attached to the ends of a thin string which passes over three smooth pegs in a wall arranged in the form of an isosceles triangle with one side horizontal and vertical angle 90°. Show that the pressure on the top peg is $\sqrt{2}\,P$ Newton and that on each of the other two pegs is $P\,(2-\sqrt{2})$ Newton.

65. Five equal forces act on a particle in the directions away from the particle so that the angle between them in order are equal. Find their resultant.

[Ans. 0]

66. OB is a diagonal of the square $OABC$. D and E are the middle points of AB and BC respectively. Find the resultant of forces P acting along OB, Q acting along OD, Q acting along OD and Q acting along OE.

$$\left[\text{Ans. } P+\frac{3}{5}\sqrt{10}Q \text{ along } OB\right]$$

67. The sides AB, BC, CD and DA of a quadrilateral $ABCD$ are bisected at E, F, G and H respectively. Show that the resultant of the forces acting at a point, which are represented in magnitude and direction by $E\,G$ and HF, is represented in magnitude and direction by AC.

68. E and F are the mid-points of the diagonals AC and BD of a quadrilateral $ABCD$. Prove that if G is the mid-point of EF, then $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}+\overrightarrow{GD}=\vec{0}$.

69. $ABCDE$ is a regular pentagon and forces acting at a point are represented by $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$, $\overrightarrow{BE}$, $\overrightarrow{CD}$, $\overrightarrow{CE}$ and $\overrightarrow{DE}$. Prove that their resultant is represented by $4\,\overrightarrow{AE}+2\,\overrightarrow{BD}$.

[*C.U.* '39]

70. Four smooth pegs A, B, C, D are fixed in the same vertical plane so that they form the four lower corners of a regular hexagon with the side BC horizontal. A string is tied to A, passes under B and C, over D and A and has a weight of $4\ N$ attached to the free end so that the weight hangs vertically. Find the pressure on the four pegs.

[Ans. $4\sqrt{4+\sqrt{3}}\ N$ on A;
$4N$ on B and C
and $4\sqrt{3}\ N$ on D]

71. $ABCDEF$ is a regular hexagon. Show that the resultant of forces $\overrightarrow{AB}+2\,\overrightarrow{AC}+3\,\overrightarrow{AD}+4\,\overrightarrow{AE}+5\,\overrightarrow{AF}$ is $\sqrt{351}\ AB$ in magnitude and the resultant is inclined at an angle of $\tan^{-1}\left(\dfrac{7}{\sqrt{3}}\right)$ with AB. [M.T.]

72. $ABCDEF$ is a regular hexagon and O is the centre of the circle which circumscribes it. Forces $4P$, $5P$, $8P$, P, $7P$ and $6P$ act along OA, OB, OC, OD, OE and OF respectively. Find the magnitude and direction of their resultant.

[Ans. P, acting along OA]

73. Two chords, OA and OB, of a circle represent, in magnitude and direction, two forces acting at the point O. Show that if their resultant passes through the centre of the circle, either the chords are equal or they contain a right angle.

74. Eight points are taken on the circumference of a circle at equal distances, and from one of the points, straight lines are drawn to the others. If these lines represent, in magnitude and direction, forces acting

on a particle at the point, show that the direction of the resultant coincides with the diameter through the point and its magnitude is four times the diameter.

75. Two constant equal forces act at the centre C of an ellipse parallel to SP and PH, where P is any point on the ellipse and S and H are the foci. Show that the end of the straight line which represents their resultant lies on a circle which passes through C.

Chapter III

Conditions of Equilibrium of Concurrent Forces

3.1 Equilibrium of Concurrent Forces

We shall now consider the conditions under which two or more forces acting at a point of a rigid body are in equilibrium i.e. the resultant of the component forces is zero.

It is obvious that a single non-zero force acting at a point of a rigid body can not maintian it in equilibrium. Hence, at least two forces are needed to maintain equilibrium. We have shown that two forces acting at a point of a rigid body will be in equilibrium only when their magnitudes are equal and they act along the same straight line but in the opposite directions. We shall now discuss the conditions of equilibrium of three forces acting at a point of a rigid body. It can be mentioned here that three forces acting on a body are in equilibrium only if they are coplanar. This has been proved in chapter VII, section 7.7.

3.2 Triangle of forces

This theorem states that *if three forces acting at a point be represented in magnitude and direction (but not in position) by the sides of a triangle, taken in order, then they will be in equilibrium.*

Let the three forces P, Q and R act at a point O and let them be represented in magnitude and direction by the sides AB, BC and CA of the triangle ABC respectively (fig. 24). To show that the three forces acting at O are in equilibrium.

We complete the parallelogram $ABCD$. Since $\overrightarrow{AD}=\overrightarrow{BC}$, hence $\overrightarrow{AD}$ repressents the force Q. Then by the parallellogram law of forces,

$$\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC},$$

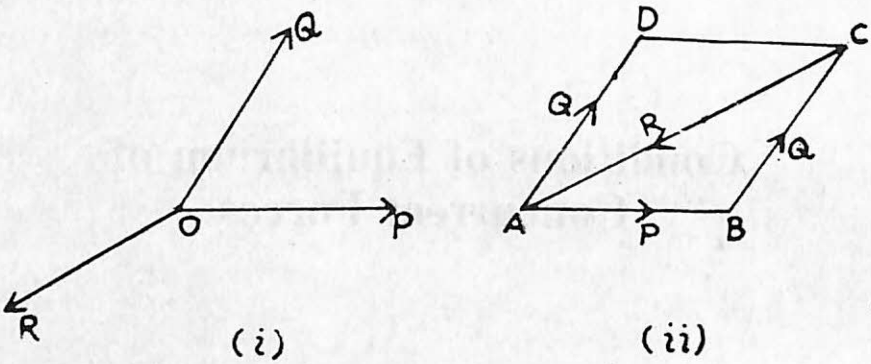


Fig. 24

Hence, the resultant of P and Q acting at O is represented by $\overrightarrow{AC}$.

Since the force R acting at O is represented by $\overrightarrow{CA}$, hence the resultant of P and Q represented by $\overrightarrow{AC}$ will balance the force R represented by $\overrightarrow{CA}$. Hence the three forces P,Q,R acting at O are in equilibrium.

Note that the same result can also be derived by using the triangle law of vectors. Thus

$$\vec{P}+\vec{Q}+\vec{R}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$$

$$=\overrightarrow{AC}+\overrightarrow{CA}=0.$$

Hence, P,Q,R are in equilibrium.

Remark 1: The above theorem implies that if the three forces P, Q and R are proportional to the sides AB, BC and CA of $\triangle ABC$, then also the forces P,Q and R acting at a point O will be in equilibrium.

Remark 2: If the three forces P, Q and R actually act along the sides of the triangle ABC, taken in order, and are represented in magnitude by those sides, then they will not be in equilibrium, but will form a couple. This has been discussed in the latter chapter VI, section 6.9. Obviously, the forces in this case will not be concurrent.

3.3 Perpendicular triangle of forces

The triangle of forces as stated above can be slightly modified and can be stated as follows.

If three forces acting at a point be such that they are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium. This is called perpendicular triangle of forces.

If we rotate the triangle through one right angle in its own plane in the proper sense, we get a triangle whose sides, in order, are parallel to the given forces and the forces are proportional to the corresponding sides of the triangle. Hence, the forces are in equilibrium.

The converse of this theorem, is also true. Thus, if three forces acting perpendicularly to the sides of a triangle are in equilibrium, they will be proportional to the sides to which they are perpendicular.

Remark: If the direction of the forces, instead of being perpendicular to the corresponding sides, make any equal angles with them, measured the same way round, then also the result will be true. The proof is exactly similar.

3.4 The Converse of the triangle of forces

This states that *if three forces acting at a point be in equilibrium, then they can be represented in magnitude and direction by the sides of a triangle, taken in order, drawn parallel to the forces.*

Let the three forces P, Q and R acting at O be in equilibrium (fig. 24). Choosing a proper scale, we draw AB and AD so that they represent P and Q in magnitude and direction. We complete the parallelogram $ABCD$ and join AC. Now, $\vec{AD}=\vec{BC}$ and so $\vec{AD}$ represents the forces Q. Now, by the parallelogram of forces,

$$\vec{AB}+\vec{AD}=\vec{AC}.$$

Hence, the resultant of P and Q acting at O can be represented by the side AC. Since the forces, P,Q and R acting at O are in equilibrium, hence the resultant force R must be equal and opposite to the resultant force $\vec{AC}$. Hence, $\vec{CA}$ must represent the force R.

Thus we get a triangle ABC whose sides AB, BC and CA, taken in order, represent the forces P, Q and R in this case. This proves the theorem.

Remark 1: If three forces acting at a point be such that the sum of the magnitude of any two is less than the magnitude of the third, then they can never be in equilibrium, for in that case, they can never be represented by the sides of any triangle.

Remark 2: If three forces acting at a point be in equilibrium, then they can also be represented in magnitude and direction by any triangle similar and similarly placed to $\triangle ABC$.

3.5 Polygon of forces

If any number of forces acting at a point be represented in magnitude and direction (but not in position) by the sides of a closed polygon, taken in order, they will be in equilibrium.

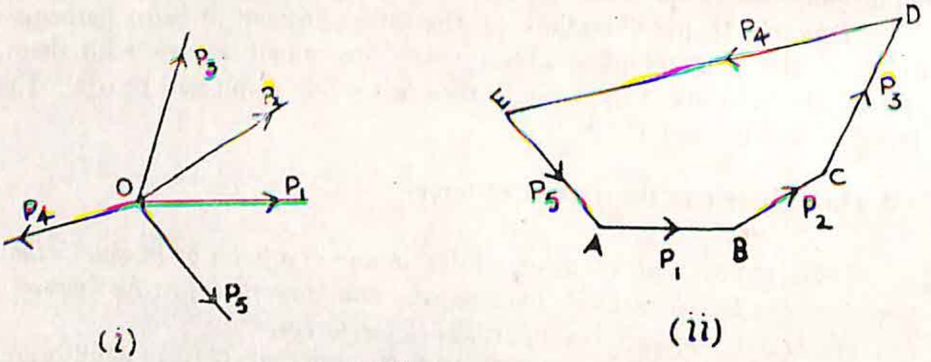


Fig. 25

Let the five forces P_1, P_2, P_3, P_4 and P_5, for example, act at O and let them be represented by $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$ and $\overrightarrow{EA}$ respectively of a polygon $ABCDE$. To prove that these forces are in equilibrium, we join AC and AD.

We have, by vector law of addition,

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$$\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$$

$$\overrightarrow{AD}+\overrightarrow{DE}=\overrightarrow{AE}$$

Hence, adding all these equations, we get

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}=\overrightarrow{AE}$$

$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EA}=\overrightarrow{AE}+\overrightarrow{EA}=0.$$

Hence, the system of forces is in equilibrium. This result is true for any number of forces.

Remark 1: The theorem is true even if the system of forces does not lie in one plane. The proof is similar.

Remark 2: As in the case of the triangle of forces, the theorem does not hold if the forces actually act along the sides of the polygon as will be shown later.

3.6 The Converse of the polygon of forces

The converse of the polygon of forces, if stated as follows, is true. *If any number of forces acting at a point be in equilibrium, then they can be represented in magnitude and direction by the sides, taken in order, of a closed polygon drawn parallel to the forces.*

We consider, as an example, five forces P_1, P_2, P_3, P_4 and P_5 acting at a point O (fig. 25). Let these forces be in equilibrium.

Let $\vec{AB}$, $\vec{BC}$, $\vec{CD}$ and $\vec{DE}$ represent the forces P_1, P_2, P_3 and P_4. To show that P_5 is represented by $\vec{EA}$.

We have shown in section 2.5 that

$$\vec{AB}+\vec{BC}+\vec{CD}+\vec{DE}=\vec{AE}$$

But since the forces are in equilibrium, hence

$$\vec{AB}+\vec{BC}+\vec{CD}+\vec{DE}+\vec{P_5}=0$$

$$\Rightarrow \vec{AB}+\vec{BC}+\vec{CD}+\vec{DE}=-\vec{P_5}$$

Hence, $\vec{AE}=-\vec{P_5}$

$$\Rightarrow \vec{P_5}=\vec{EA}.$$

This completes the proof of the theorem. This result is true for any number of forces.

Remark 1: The forces P_1, P_2, P_3,... need not be coplanar.

Remark 2: It should be noted here that if a number of forces acting at a point be in equilibrium, then they cannon necessarily be represented by the sides of any polygon drawn with its sides parallel to the forces, because polygons with their corresponding sides parallel are not necessarily similar i.e. their corresponding sides need no be always proportional.

3.7 Lami's Theorem

Father Lami gave the triangle of forces a trigonometrical form in 1687. This is known as Lami's Theorem which can be stated as follows:

If three forces acting at a point be in equilibrium, then each force is proportional to the sine of the angle between the other two forces.

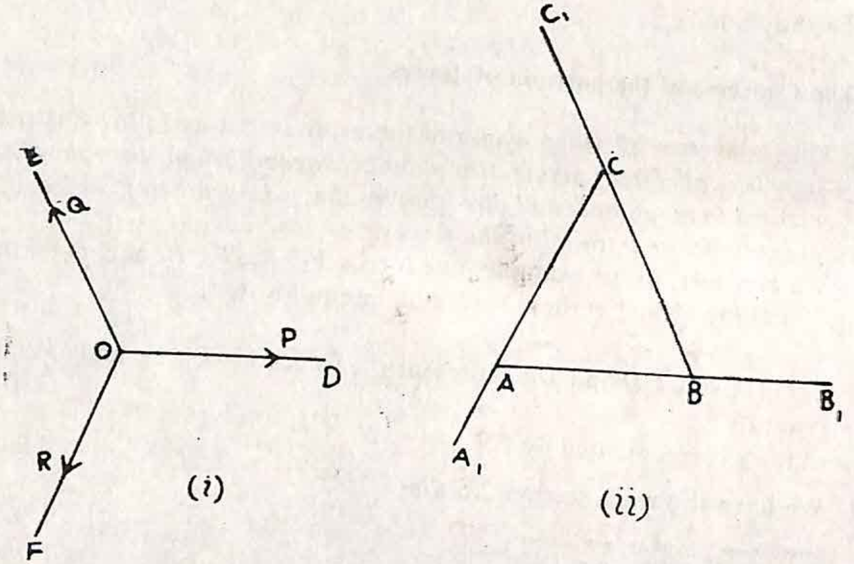


Fig. 26

Let the three forces P, Q and R acting at a point O be in equilibrium. Hence, by the converse of triangle of forces, they can be represented by the sides of a triangle ABC, taken in order, whose sides are respectively parallel to the forces.

Let OD, OE, and OF be respectively parallel to AB, BC and CA. We produce AB, BC and CA to B_1, C_1 and A_1 respectively so that

$\angle EOD = \angle CBB_1$, $\angle EOF = \angle ACC_1$
and $\angle FOD = \angle BAA_1$.

$$\text{Hence,} \quad \frac{P}{AB} = \frac{Q}{BC} = \frac{R}{AC} \qquad \ldots(1)$$

$$\text{But} \quad \frac{AB}{\sin \angle ACB} = \frac{BC}{\sin \angle CAB} = \frac{CA}{\sin \angle ABC}$$

$$\Rightarrow \frac{AB}{\sin\,(180^\circ - \angle ACC_1)} = \frac{BC}{\sin\,(180^\circ - \angle BAA_1)}$$

$$= \frac{CA}{\sin\,(180^\circ - \angle CBB_1)}$$

$$\Rightarrow \frac{AB}{\sin \angle ACC_1} = \frac{BC}{\sin \angle BAA_1} = \frac{CA}{\sin \angle CBB_1}$$

$$\Rightarrow \frac{AB}{\sin \angle EOF} = \frac{BC}{\sin \angle FOD} = \frac{CA}{\sin \angle EOD}$$

$\therefore$ From (1),

$$\frac{P}{\sin \angle EOF} = \frac{Q}{\sin \angle FOD} = \frac{R}{\sin \angle EOD}$$

i.e. P, Q and R are proportional to the sine of the angle between the other two forces.

Alternative Proof :

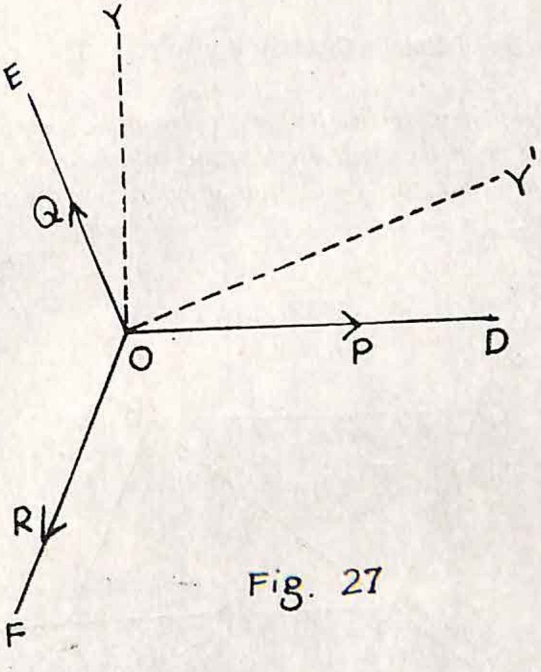


Fig. 27

Let the forces P, Q and R acting at a point O in the directions OD, OE and OF respectively be in equilibrium. Hence, the three forces are coplanar. Let OY be perpendicular to OD and OY' be perpendicular to OE. Since the resultant of the three forces P, Q and R is zero, the algebraic sum of their resolved parts in any direction, being equal to the resolved part of their resultant in the same direction, is zero. We now resolve these forces along OY and OY'. Hence, we have

$Q \sin \angle EOD + R \sin (360° - \angle FOD) = O$

$\Rightarrow Q \sin \angle EOD = R \sin \angle FOD$ .. (1)

and $P \sin \angle EOD + R \sin (360° - \angle EOF) = 0$

$\Rightarrow P \sin \angle EOD = R \sin \angle EOF$ ...(2)

From (2), we have $\dfrac{P}{\sin \angle EOF} = \dfrac{R}{\sin \angle EOD}$

and from (1), we have $\dfrac{R}{\sin \angle EOD} = \dfrac{Q}{\sin \angle FOD}$.

Hence, we get $\dfrac{P}{\sin \angle EOF} = \dfrac{Q}{\sin \angle FOD} = \dfrac{R}{\sin \angle EOD}$.

3.8 The Converse of Lami's Theorem

The converse of Lami's theorem is also true. This can be stated as follows:

If three coplanar forces acting at a point be such that each is proportional to the sine of the angle between the other two (the sense of the forces being such that the equilibrium is not absurd), then the forces will be in equilibrium.

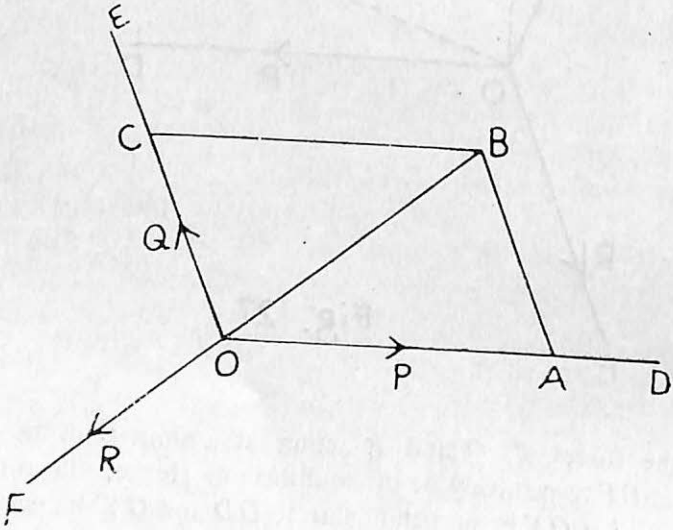


Fig. 28

Let the forces $P.Q$ and R act at O along OD, OE and OF. With a chosen scale, we cut off a length OA to represent the force P. We draw a line AB through A parallel to OE. We produce OF to meet AB at B. From $\triangle\ OAB$, we have

$$\frac{OA}{\sin\angle OBA}=\frac{AB}{\sin\angle BOA}=\frac{OB}{\sin\angle BAO}$$

$$\Rightarrow \frac{P}{\sin(180°-\angle COF)}=\frac{AB}{\sin(180°-\angle AOF)}=\frac{OB}{\sin(180°-\angle COA)}$$

$$\Rightarrow \frac{P}{\sin\angle COF}=\frac{AB}{\sin\angle AOF}=\frac{OB}{\sin\angle COA}.$$

Now, it is given that

$$\frac{P}{\sin\angle COF}=\frac{Q}{\sin\angle AOF}=\frac{R}{\sin\angle COA}.$$

Hence, $AB=Q$ and $OB=R$ on the same scale as $OA=P$.

Also by construction, $OA=P$. Hence, the sides of the triangle OAB, taken in order, represent the forces P,Q and R in magnitude and direction. Hence, by the triangle of forces, these forces are in equilibrium.

3.9 Conditions of equilibrium of a number of a coplanar and concurrent forces

We have shown in chapter II, section 2.5 that the resultant R of a number of forces $P_1, P_2, \ldots\ldots P_n$ is given by

$R=\sqrt{X^2+Y^2}$,

where X is the algebraic sum of the resolved parts of all the forces along the x-axis and Y is that along the y-axis. If all the forces are in equilibrium, then R must be zero.

Hence, $X=0$ and $Y=0$.

Conversely, if $X=X$ and $Y=0$, then R must be zero and hence the system of forces will be in equilibrium. We observe that the x-axis can be chosen arbitrarily in the plane of the forces. Thus we have the following theorem on the equilibrium of a number of coplanar aud concurrent forces:

Theorem : The necessary and sufficient conditions that a system of coplanar forces acting at a point may be in equilibrium are that the algebraic sum of the resolved parts af the forces along any two mutually perpendicular directions or any two arbitrary directions should be separately zero.

3.10 Equilibrium of a system of particles

When several particles are connected together and form a system and each particle is acted upon by special forces in addition to the forces produced upon it by its connection by strings or rods with the other particles, we can consider the equilibrium of any one particle apart from all the others, provided that we consider all the forces which are produced on it by its connection with the others, in addition to the special forces acting on it. This is called the *principle of separate equilibrium.*

Thus in Example 8 of section 3.11, we can write down the equations for the equilibrium of the particle A_3 as if it were entirely disconnected with the other points A_1, A_2, A_4,......, if we represent it as acted on by the vertically downward force W_2 and the tensions T_2 and T_3 of the strings by which it is connected with the system.

3.11 Solved Examples

Ex. 1. A weight of 26 N is suspended by two light inelastic strings of lengths 5m and 12m from two points at the same level and 14m apart. Find the tensions in the strings.

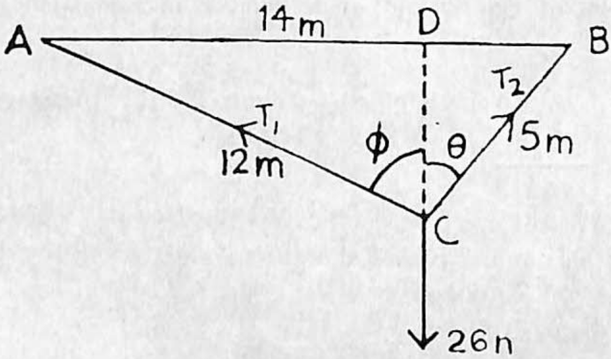


Fig. 29

Let the weight of 26N be suspended at C by two strings CA and CB. AB=14m, AC=12m and CB=5 m. Let T_1 N and T_2 N be the tensions of the strings CA and CB. Every point of the string, say AC, is pulled by two equal forces T_1, T_1 in opposite directions and remains in equilibrium. At C, the tension T_1 acts along CA while at A, this acts along AC. Similarly, the tension T_2 of the string CB acts along CB for the point C and along BC for the point B.

Let $CD \perp AB$ and let $\angle DCB = \theta$, $\angle DCA = \phi$.

The particle at C is in equilibrium under the coplanar forces $T_1 N$ along CA, $T_2 N$ along CB and $26N$ vertically downwards.

Hence, by Lami's theorem, we have

$$\frac{T_1}{\sin \theta} = \frac{T_2}{\sin \phi} = \frac{26}{\sin (\theta + \phi)}$$

Now, $\sin \theta = \sin (90° - B) = \cos B$,

$\sin \phi = \sin (90° - A) = \cos A$ and

from $\triangle ABC$,

$$AC^2 = AB^2 + BC^2 - 2AB.BC \cos B$$
$$\Rightarrow 12^2 = 14^2 + 5^2 - 2 \times 14 \times 5 \cos B$$
$$\Rightarrow 140 \cos B = 77 \Rightarrow \cos B = \frac{11}{20}$$

Also,

$$BC^2 = AB^2 + AC^2 - 2AB.\ AC \cos A$$
$$\Rightarrow 5^2 = 14^4 + 12^2 - 2 \times 14 \times 12 \cos A$$
$$\Rightarrow 336 \cos A = 315$$
$$\Rightarrow \cos A = \frac{15}{16}.$$

Hence, $\sin \theta = \cos B = \frac{11}{20}$,

$\sin \phi = \cos A = \frac{15}{16}$

and $\sin (\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$

$$= \frac{11}{20} \times \sqrt{1 - \frac{225}{256}} + \sqrt{1 - \frac{121}{400}} \times \frac{15}{16}$$
$$= \frac{11}{20} \times \frac{\sqrt{31}}{16} + \frac{3\sqrt{31}}{20} \times \frac{15}{16} = \frac{7\sqrt{31}}{40}$$

Hence, $\dfrac{T_1}{\frac{11}{20}} = \dfrac{T_2}{\frac{15}{16}} = \dfrac{26}{\frac{7\sqrt{31}}{40}}$

$$\therefore\ T_1 = \frac{572}{7\sqrt{31}},\ T_2 = \frac{975}{7\sqrt{31}}.$$

Hence, the tensions of the strings are

$$\frac{572}{7\sqrt{31}} N \text{ and } \frac{975}{7\sqrt{31}} N.$$

Ex. 2. Three forces in equilibrium act perpendicularly to the sides of a triangle through any point in their plane within the triangle. Show that the forces are proportional to the corresponding sides of the triangle.

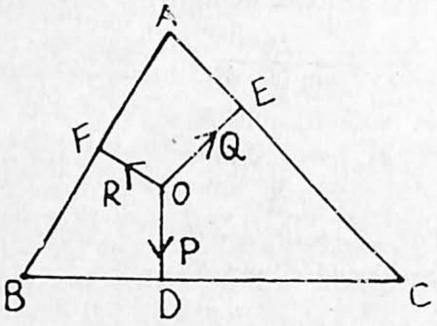


Fig. 30

Let the forces P, Q and R act at a point O within the triangle ABC along OD, OE and OF repectively where $OD \perp BC$, $OE \perp AC$ and $OF \perp AB$. It is given that the forces are in equilibrium. Hence, by Lami's theorem, we have

$$\frac{P}{\sin \angle FOE} = \frac{Q}{\sin \angle FOD} = \frac{R}{\sin \angle EOD}$$

$$\Rightarrow \frac{P}{\sin (180° - A)} = \frac{Q}{\sin (180° - B)} = \frac{R}{\sin (180° - C)},$$

since the quadrilaterals $AFOE$, $FODB$, $DOEC$ are cyclic.

Hence, $\dfrac{P}{\sin A} = \dfrac{Q}{\sin B} = \dfrac{R}{\sin C}$,

$$\Rightarrow \frac{P}{a} = \frac{Q}{b} = \frac{R}{c},$$

since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, where

a,b,c, are the sides of the triangle ABC.

Ex. 3. A force P acting along a smooth inclinced plane can support a weight W and acting horizontally can support a weight W on the same piane. Prove that $P^2 = W_1^2 - W_2^2$.

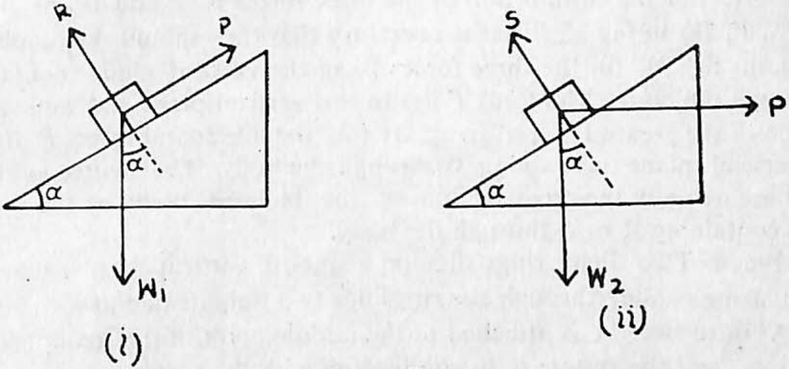


Fig. 31

Let R be the reaction of the plane on the body resting on the inclined plane [fig. 31 (i)] of inclination α when the force P acts along the plane. Let S be the reaction of the same plane when the force P acts horizontally [fig. 31(ii)]. In both the cases the reactions of the plane on the bodies will be perpendicular to the plane, since the plane is smooth.

In the first case, the body is in equilibrium under the forces R, P and W_1 which act at a point on the body. Hence, by Lami's theorem, we have

$$\frac{P}{\sin(180^\circ-\alpha)}=\frac{W_1}{\sin 90^\circ}=\frac{R}{\sin(90^\circ+\alpha)}$$

$$\Rightarrow \frac{P}{\sin\alpha}=\frac{W_1}{1}=\frac{R}{\cos\alpha} \qquad \ldots(1)$$

In the second case, the three forces P, S and W_2 acting at a point of the body are in equilibrium. Hence, by Lami's theorem, we have

$$\frac{P}{\sin(180^\circ-\alpha)}=\frac{W_2}{\sin(90^\circ+\alpha)}=\frac{S}{\sin 90^\circ}$$

$$\Rightarrow \frac{P}{\sin\alpha}=\frac{W_2}{\cos\alpha}=\frac{S}{1} \qquad \ldots(2)$$

From (1), we have $W_1=P\operatorname{cosec}\alpha$

and from (2), $W_2=P\cot\alpha$.

Hence, $W^2_1-W^2_2=P^2(\operatorname{cosec}^2\alpha-\cot^2\alpha)=P^2$.

Alternative method:

Resolving all the forces along the plane in the two cases, we get

$P=W_1\sin\alpha\Rightarrow W_1=P\operatorname{cosec}\alpha$

and $P\cos\alpha=W_2\sin\alpha\Rightarrow W_2=P\cot\alpha$.

Hence, $W^2_1-W^2_2=P^2(\operatorname{cosec}^2\alpha-\cot^2\alpha)=P^2$

Note: For the equilibrium of the three forces R, P and W_1 in fig. 31 (i) or S, P, W_2 in fiig 32 (ii), it is necessary that they should be coplanar. Hence, in fig 31 (i), the three forces lie in the vertical plane containing R through the body. The force P lies in this vertical plane and acts along the line of the greatest slope. In fig. 31 (ii), the horizontal force P lies in the vertical plane containing S through the body. The figures as shown above are actually the cross-section of the inclined plane by the vertical plane containing R or S through the body.

Ex. 4. Two light rings slide on a smooth vertical circular wire and a thin string passing through the rings has two weights tied at its extremities. A third weight is attached to the middle point of the string between the rings, and the system is in equilibrium with the rings resting at points such that each of the radii through P and Q makes an angle of 30° with the vertical through the centre. Find the relation between the weights suspended. Find also the pressure on the wire at one ring, if the middle weight be 10 N.

Let O be the centre of the circular wire and AO the vertical through O. Let P and Q be the positions of the rings on the opposite sides of the vertical AO such that $\angle AOP = \angle AOQ = 30°$.

Let W_1 and W_2 be the weights suspended at P and Q acting vertically downwards and let W be the weight suspended at B, the middle point of the string PBQ, acting vertically downwards. Let R_1 and R$_2$ be the reactions of the wire on the rings at P and Q respectively. Since the wire is smooth, its reaction on the rings will be normal to the circle and hence the lines of action of R_1 and R_2 will pass through the centre of the circle. Since the weight W is attached with the middle point B of the string, the tensions of the two portions of the strings on the two sides of B will be, in general, different. Let T_1 and T_2 be the tensions of the strings passing through the rings P and Q respectively. Let U and V be any two points below P and Q respectively on the vertical lines through P and Q (not shown in the figure). Since the rings are smooth, the tension of the vertical portion PU of the string through P and the portion PB of the string will be the same i.e. T_1. Similarly, the tension of the portion BQ of the string and the remaining vertical portion QV of the string through Q will also be the same i.e. T_2.

Now, the load W_1 is balanced by the tension T_1 of the string.

Hence, $T_1 = W_1$.

Similarly, considering the equilibrium of the load W_2, we have $T_2 = W_2$.

Now; since $PB = QB$, we have $\angle PBA = \angle QBA$.

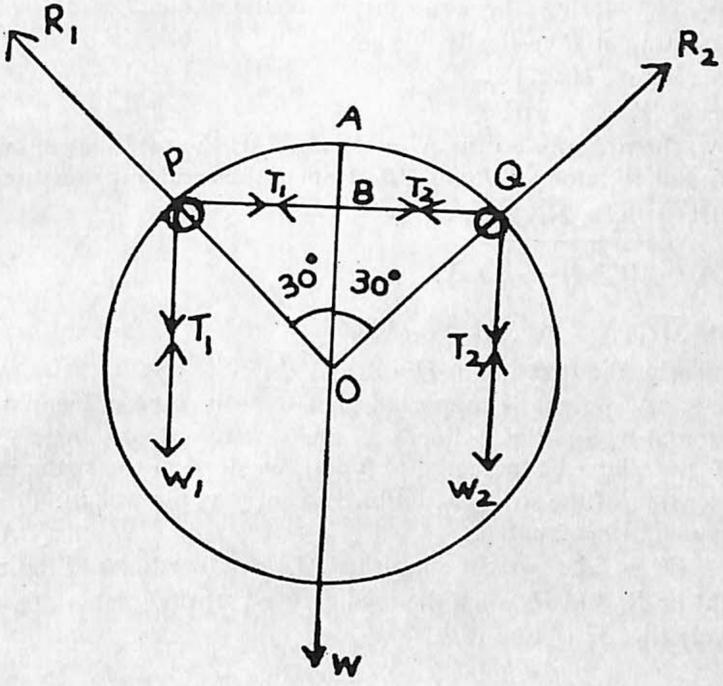


Fig. 32

Now, the point B is in equilibrium under the forces T_1 along BP, T_2 along BQ and W acting vertically along BO. Resolving these forces horizontally, we get

$T_1 \sin \angle PBA = T_2 \sin \angle QBA$

$\Rightarrow T_1 = T_2$

$\Rightarrow W_1 = W_2$.

Now, the ring at P is in equilibrium under the forces $T_1 = W_1$ along PB, W_1 along PU and R_1 along OP.

Now, resolving the forces acting at P perpendicular to OP, we get

$W_1 \sin \angle UPO = W_1 \sin \angle OPB$

$\Rightarrow \angle UPO = \angle OPB$

But $\angle UPO = \angle POA = 30^\circ$

$\therefore \angle UPO = \angle OPB = 30^\circ$

$\therefore \angle UPB = 30^\circ + 30^\circ = 60^\circ$

$\therefore \angle ABP = \angle ABQ = 60^\circ$.

Now, considering the equilibrium of the point B, and resolving all the forces acting at B vertically, we get

$W=2W_1 \cos 60° = W_1$.

Hence, $W_1=W_2=W$.

Now, the pressure on the wire at, say P, is the resultant of two equal forces W_1 and W_1 along PU and PB. Hence, the resultant pressure at P is

$$\sqrt{W_1^2+W_1^2+2W_1^2 \cos 60°}\ N$$

$$=\sqrt{2W_1^2\ (1+\frac{1}{2})}\ N.$$

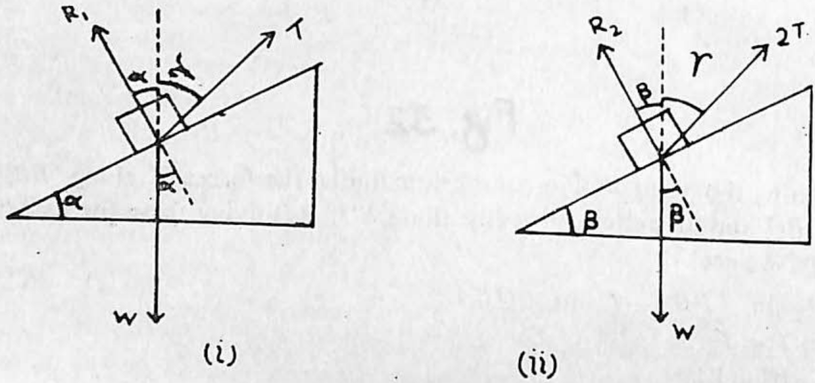
$$=\sqrt{3W_1^2}\ N=\sqrt{3}\ W_1\ N=10\sqrt{3}\ N.$$

Similarly, the pressure at $Q=10\sqrt{3}\ N$.

Ex. 5. A weight is supported on a smooth plane of inclination α to the horizontal by a string inclined to the vertical at an angle γ. If the slope of the plane be increased to β and the slope of the string is unaltered, the tension of the string is doubled to support the weight. Prove that

$\cot \alpha - \cot \gamma = 2 \cot \beta$. [C.U. 1945]

Let W be the weight supported. Let the reactions of the plane on the weight be R_1 and R_2 when the inclinations of the plane are α and β respectively (fig. 32 (i) and (ii)).



(i) (ii)

Fig. 33

Since the plane is smooth, hence its reactions on the weight in the two cases will be perpendicular to the plane. Let the tensions of the string be T and $2T$ in the two cases as shown in the figures. In the first case, the direction of T makes an angle of $\alpha+\gamma$ with the normal to the plane and in the second case the direction of $2T$ makes an angls of $\beta+\gamma$ with the normal to the plane. In the first case, the body is in equilibrium under its weight W acting vertically downwards, the reaction R_1 acting perpendicular to the plane and the tension T of the string. In the second case, the body is in equilibrium under its weight W acting vertically downwards, the reaction R_2 acting perpendicular to the plane and the tension $2T$.

Resolving the forces along the plane in the two cases, we get

$$T \sin (\alpha+\gamma)=W \sin \alpha \qquad \ldots(1)$$

and $2T \sin (\beta+\gamma)=W \sin \beta \qquad \ldots(2)$

Dividing (1) by (2), we get

$$\frac{\sin (\alpha+\gamma)}{2 \sin (\beta+\gamma)}=\frac{\sin \alpha}{\sin \beta}$$

$$\Rightarrow \sin (\alpha+\gamma) \sin \beta=2 \sin (\beta+\gamma) \sin \alpha$$

$$\Rightarrow \sin \beta\,(\sin \alpha \cos \gamma+\cos \alpha \sin \gamma)$$

$$=2 \sin \alpha\,(\sin \beta \cos \gamma+\cos \beta \sin \gamma)$$

$$\Rightarrow \sin \alpha \sin \beta \cos \gamma+\sin \beta \cos \alpha \sin \gamma$$

$$=2 \sin \alpha \sin \beta \cos \gamma+2 \sin \alpha \cos \beta \sin \gamma$$

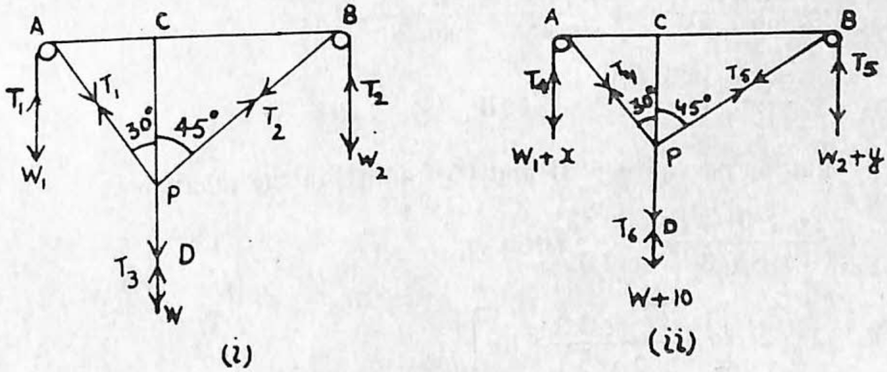
$$\Rightarrow \sin \beta \cos \alpha \sin \gamma-\sin \alpha \sin \beta \cos \gamma=2 \sin \alpha \cos \beta \sin \gamma$$

Dividing both sides by $\sin \alpha \sin \beta \sin \gamma$,

we get

$$\cot \alpha-\cot \gamma=2 \cot \beta.$$

Ex. 6. Three strings are attached to a light particle P which is in equilibrium. Two of them pass over pulleys and then hang vertically, carrying weights, while the third supports a weight WN. The inclinations of the upper strings are respectively 30° and 45° to the upward vertical through P. An additional weight of 10 N is now attached to W. Find what additional weights must be added to the other two strings to ensure that P shall remain in equilibrium in the same position.



(i) (ii)

Fig. 34

Let the weights suspended from the ends of three strings PA, PB and PD be W_1, W_2 and W respectively [fig. 34 (i)]. Let x N and y N be the additional weights which are attached with the weights W_1 and W_2 respectively when an additional weights of 10 N is attached to W [fig. 34 (ii)]. Let T_1, T_2, and T_3 be the tensions of the strings PA, PB and PD respectively in the first case and T_4, T_5, T_6 be those of the strings in the second case. In both the cases, the particle P is in equilibrium. It is clear that

$$T_1 = W_1,\ T_2 = W_2,\ T_3 = W$$

$$\text{and } T_4 = W_1 + \text{x},\ T_5 = W_2 + \text{y},\ T_6 = W + 10.$$

In the fiirst case, the particle P is in equilibrium under the forces W, W_1 and W_2. Hence, by applying Lami's theorem, we get

$$\frac{W}{\sin(30^\circ + 45^\circ)} = \frac{W_1}{\sin 45^\circ} = \frac{W_2}{\sin 30^\circ}$$

$$\Rightarrow \frac{W}{\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}} = \frac{W_1}{\frac{1}{\sqrt{2}}} = \frac{W_2}{\frac{1}{2}}$$

$$\Rightarrow \frac{2\sqrt{2}W}{1+\sqrt{3}} = \sqrt{2}\, W_1 = 2W_2 = K, \text{ say.}$$

$$\therefore W = \frac{(\sqrt{3}+1)K}{2\sqrt{2}},\ W_1 = \frac{K}{\sqrt{2}},\ W_2 = \frac{K}{2}. \qquad \ldots(1)$$

In the second case, the particle P is in equilibrium under the forces $W_1 + \text{x}$ and $W_2 + \text{y}$. Hence, by applying Lami's theorem, we get

$$\frac{W+10}{\sin(30^\circ + 45^\circ)} = \frac{W_1 + \text{x}}{\sin 45^\circ} = \frac{W_2 + \text{y}}{\sin 30^\circ}$$

$$\Rightarrow \frac{2\sqrt{2}\,(W+10)}{1+\sqrt{3}} = \sqrt{2}\,(W_1 + \text{x}) = 2\,(W_2 + \text{y})$$

Putting the values of W and W_1 from (1) in the equation

$$\frac{2\sqrt{2}\,(W+10)}{1+\sqrt{3}} = \sqrt{2}\,(W_1 + \text{x}), \text{ we get}$$

$$\frac{2\sqrt{2}\left[\frac{(\sqrt{3}+1)K}{2\sqrt{2}} + 10\right]}{1+\sqrt{3}} = \text{K} + \sqrt{2}\,\text{x}$$

$$\Rightarrow \frac{(\sqrt{3}+1)K+20\sqrt{2}}{1+\sqrt{3}} = K+\sqrt{2}\,x$$

$$\Rightarrow \sqrt{2}\,(1+\sqrt{3})\,x = 20\sqrt{2}$$

$$\therefore x = \frac{20}{\sqrt{3}+1} = 10\,(\sqrt{3}-1).$$

Again, putting the values of W and W_2 from (1) in the other equation

$$\frac{2\sqrt{2}\,(W+10)}{1+\sqrt{3}} = 2\,(W_2+y), \text{ we get similarly}$$

$$\frac{(\sqrt{3}+1)K+20\sqrt{2}}{1+\sqrt{3}} = K+2y$$

$$\Rightarrow 2\,(1+\sqrt{3})\,y = 20\sqrt{2}$$

$$\Rightarrow y = \frac{10\sqrt{2}}{1+\sqrt{3}} = 5\sqrt{2}\,(\sqrt{3}-1).$$

Hence, the required additional weights are $10\,(\sqrt{3}-1)\,N$ and $5\sqrt{2}\,(\sqrt{3}-1)\,N$.

Ex. 7. If one of the two intersecting forces be given in magnitude and direction and only the line of action of the other force is given, prove that the least force which will produce equilibrium is perpendicular to the second force.

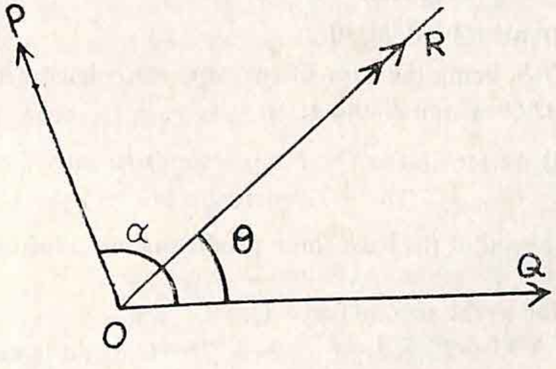


Fig. 35

Let the first and the second forces be $\vec{P}$ and $\vec{Q}$ respectively and let them intersect at O. The magnitude of the force $\vec{P}$ is given but not of $\vec{Q}$.

Since the directions of both $\vec{P}$ and $\vec{Q}$ are known, the angle between the forces is also given. Let the given angle between $\vec{P}$ and $\vec{Q}$ be α. Let $\vec{R}$ be the resultant of $\vec{P}$ and $\vec{Q}$ inclined at an angle θ with the direction of $\vec{Q}$. The third force which together with the forces $\vec{P}$ and $\vec{Q}$ will produce equilibrium will be least only if R is least, since the resultant of $\vec{P}$ and $\vec{Q}$ will be balanced by the third force for equilibrium. Hence, the direction of the least force which will produce equilibrium with the forces $\vec{P}$ and $\vec{Q}$, will be opposite to the direction of the least value of R. We, therefore, seek that value of θ for which R is the least. We have

$$\tan\theta = \frac{P\sin\alpha}{Q+P\cos\alpha} \qquad \ldots(1)$$

and $R^2 = P^2+Q^2+2PQ\cos\alpha$. $\qquad\ldots(2)$

From (2), we see that R is least if

$P^2+Q^2+2PQ\cos\alpha$ is least for given P and α.

Now, $R^2=(Q+P\cos\alpha)^2+P^2-P^2\cos^2\alpha$

$=P^2\sin^2\alpha+(Q+P\cos\alpha)^2$.

The *R.H.S.* being the sum of two squares is least when $Q+P\cos\alpha=O$ for given P and α.

From (1) we see that as $Q+P\cos\alpha$ tends to zero, θ tends to $\frac{\pi}{2}$. It, therefore, follows that the least force producing equilibrium with $\vec{P}$ and $\vec{Q}$ is perpendicular to the second force $\vec{Q}$.

Ex. 8. A string $A_1A_2A_3\ldots A_{n+2}$ whose weight is neglected is suspended from two fixed points A_1 and A_{n+2} and from the points A_2, $A_3, \ldots A_{n+1}$ in the string are suspended a series of particles of weights $W_1, W_2, W_3, \ldots W_n$ respectively. If the inclination of the successive portions of the string to the horizon are $\theta_1, \theta_2, \ldots \theta_{n+1}$, show that the horizontal components of the tensions of different parts of the string are all equal.

If $W_1 = W_2 = W_3 = \ldots\ldots = W_n = W$, then prove that the tangents of the successive inclinations of the portions of the string form a series in *A.P.*

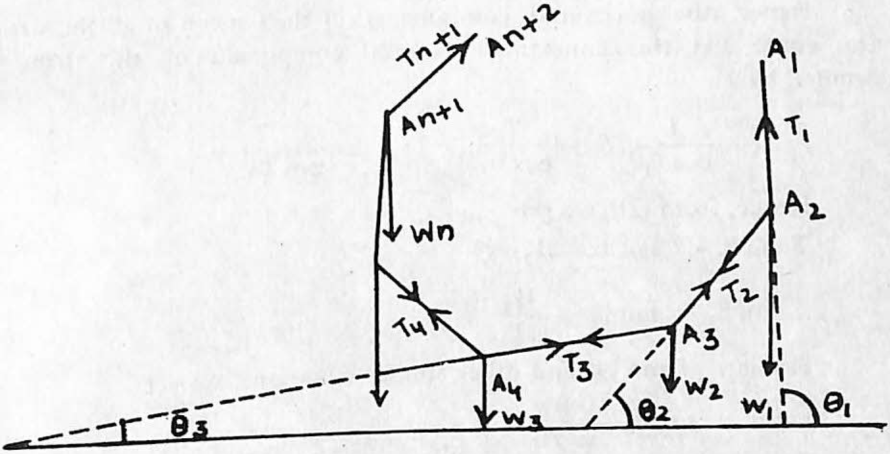


Fig. 36

The tensions of different portions of the string due to the attachment of different weights will be different. Let T_1, T_2, T_3,……T_{n+1} be the tensions of different parts of the string.

Considering the equilibrium of the particle at A_2, we see that it is in equilibrium under the forces T acting along A_2A_1, T_2 acting along $A_2 A_3$ and W_1 acting vertically downwards.

Hence, resolving all the forces horizontally, we get

$$T_1 \cos \theta_1 = T_2 \cos \theta_2 \qquad \ldots(1)$$

Now, resolving the forces vertically, we get

$$T_1 \sin \theta_1 - T_2 \sin \theta_2 - W_1 = O \qquad \ldots(2)$$

Similarly, the particle at A_3 is in equilibrium under the forces T_2 acting along $A_3 A_2$, T_3 acting along $A_3 A_4$ and W_2 acting vertically downwards. Hence, resolving horizontally and vertically, we get

$$T_2 \cos \theta_2 = T_3 \cos \theta_3 \qquad \ldots(3)$$

$$\text{and } T_2 \sin \theta_2 - T_3 \sin \theta_3 - W_2 = O \qquad \ldots\ldots(4)$$

We shall get similar pairs of equations from the equilibrium condition of other particles.

From (1), (3) and similar other equations, we get

$T_1 \cos \theta_1 = T_2 \cos \theta_2 = T_3 \cos \theta_3 = \ldots = T_{n+1} \cos \theta_{n+1}$

Hence, the horizontal components of the tension of all the strings are equal. Let this constant horizontal component of the string be denoted by T.

$$\therefore T_1 = \frac{T}{\cos \theta_1},\ T_2 = \frac{T}{\cos \theta_2}, \ldots\ T_{n+1} = \frac{T}{\cos \theta_{n+1}}.$$

Hence, from (2), we get

$T \tan \theta_1 - T \tan \theta_2 - W_1 = O$

$$\therefore \tan \theta_1 = \tan \theta_2 + \frac{W_1}{T} \qquad \ldots(\alpha_1)$$

Similarly, from (4) and other similar equations, we get

$$\tan \theta_2 = \tan \theta_3 + \frac{W_2}{T} \qquad \ldots(\alpha_2)$$

……………………………………

$$\tan \theta_n = \tan \theta_{n+1} + \frac{W_n}{T} \qquad (\alpha_n)$$

If $W_1 = W_2 = W_3 = \ldots\ldots = W_n = W$, say, then

from (α_1), (α_2), ……(α_n), we get

$$\tan \theta_1 - \tan \theta_2 = \tan \theta_2 - \tan \theta_3 = \ldots\ldots$$

$$= \tan \theta_n - \tan \theta_{n+1} = \frac{W}{T}$$

which shows that the tangents of the successive inclinations of the portions of the string form a series in *A.P.*

Ex. 9. A smooth tube in the form of a parabola is placed with its axis vertical and vertex downwards and a heavy particle is placed within it. Show that the particle can be kept at rest by an outward force along an ordinate which varies as the ordinate and that the corresponding reaction of the tube varies as the square root of its distance from the focus of the parabola.

Let O be the vertex of the parabola which is chosen as the origin. We choose the axis of the parabola and the tangent at the vertex as the x and y axes respectively. Let the equation of the parabola be

$y^2 = 4ax$, where $a > o$.

Let P (x_1, y_1) be the position of the particle and the tangent PT at P intersect the x-axis at T. Let the inclincation of this tangent with the x-axis be θ. The normal PG at P intersects the x-axis at G. We draw PN

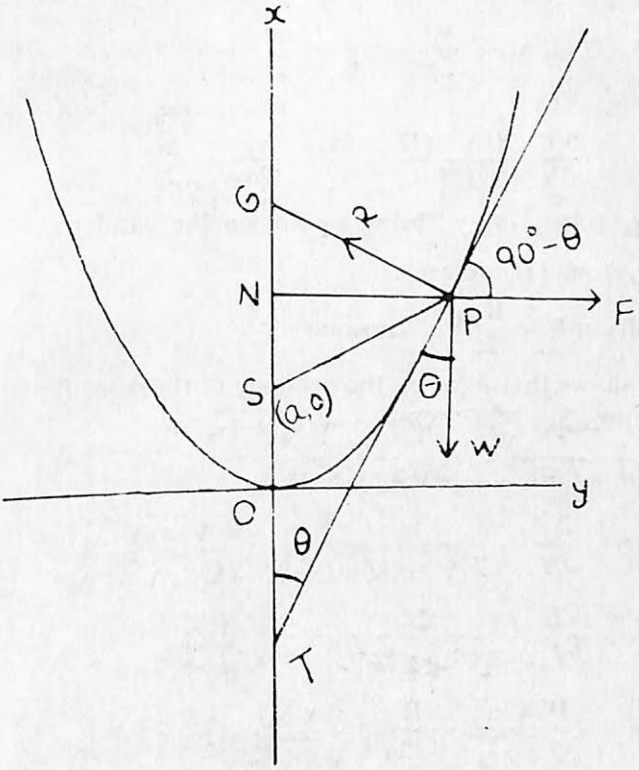


Fig 37

perpendicular to the x-axis Let S be the focus of the parabola. The particle at P is in equilibrium under its weight W acting vertically downwards, the normal reaction R of the smooth tube acting along the normal PG and the outward force F acting along the ordinate NP. Resolving all these forces along the tanget PT and the normal PG, we get

$$W \cos \theta = F \sin \theta \qquad \ldots\ldots(1)$$

and $R = W \sin \theta + F \cos \theta$ ......(2)

Now, since $y^2 = 4\ ax$, hence

$$2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

∴The equation of the tangent to the parabola at (x_1, y_1) is

$$y - y_1 = \frac{a^2}{y_1}\ (yx - x_1).$$

This tangent cuts the x-axis at the point where $y=0$. This gives

$$x = -OT = x_1 - \frac{y_1^2}{2a} = x_1 - \frac{4ax_1}{2a} = -x_1$$

$$\therefore OT = x_1 = ON.$$

$$\therefore \cot\theta = \frac{NT}{PN} = \frac{ON+OT}{PN} = \frac{2x_1}{y_1} = \frac{y_1^2}{2ay_1} = \frac{y_1}{2a},$$

since $y_1^2 = 4ax_1$, (x_1, y_1) being a point on the parabola.

Hence, from (1), we get

$$F = W\cot\theta = \frac{Wy_1}{2a} = \text{constant} \times y_1 \qquad \ldots(3)$$

which shows that $F \propto y_1$, the ordinate of the point P.

Now, $PT = \sqrt{PN^2 + NT^2} = \sqrt{y_1^2 + 4x_1^2}$

$$= \sqrt{4ax_1 + 4x_1^2} = 2\sqrt{x_1}\sqrt{a+x_1}$$

$$\therefore \sin\theta = \frac{NP}{PT} = \frac{y_1}{2\sqrt{x_1}\sqrt{a+x_1}} = \frac{2\sqrt{a}\sqrt{x_1}}{2\sqrt{x_1}\sqrt{a+x_1}} = \frac{\sqrt{a}}{\sqrt{a+x_1}}$$

$$\cos\theta = \frac{NT}{PT} = \frac{2x_1}{2\sqrt{x_1}\sqrt{a+x_1}} = \frac{\sqrt{x_1}}{\sqrt{a+x_1}}$$

$$\therefore R = \frac{W\sqrt{a}}{\sqrt{a+x_1}} + \frac{Wy_1}{2a}\cdot\frac{\sqrt{x_1}}{\sqrt{a+x_1}} \text{ [from (3)]}$$

$$= \frac{W\sqrt{a}}{\sqrt{a+x_1}} + \frac{W2\sqrt{a}\sqrt{x_1}\sqrt{x_1}}{2a\sqrt{a+x_1}}$$

$$= \frac{W\sqrt{a}}{\sqrt{a+x_1}} + \frac{Wx_1}{\sqrt{a+x_1}\sqrt{a}} = \frac{Wa+Wx_1}{\sqrt{a}\sqrt{a+x_1}} = \frac{W\sqrt{a+x_1}}{\sqrt{a}} \qquad \ldots(4)$$

Now, since the co-ordinates of the focus S of the parabola are (a,o), hence the distance PS is given by

$$PS = \sqrt{(x_1-a)^2 + y_1^2} = \sqrt{(x_1-a)^2 + 4ax_1}$$

$$= \sqrt{(x_1+a)^2}$$

$= x_1 + a$, assuming that $x_1 + a > 0$

Hence, from (4), we have

$$R = \frac{W}{\sqrt{a}}\sqrt{PS} = \text{constant} \times \sqrt{PS},$$

which shows that the reaction of the tube at P varies as the square root of its distance from the focus of the parabola.

Ex. 10. Three poles *PA*, *PB* and *PC*, each 9m. long, form a tripod from the vertex *P* of which a weight *W* is hung; the feet of the poles rest on a horizontal plane, rough enough to prevent any slipping, and form a triangle *ABC*, the length of whose sides are $BC=AC=5$m and $AB=6$m. If T_1, T_2 and T_3 be the thrusts of the poles *PA*, *PB* and *PC* respectively, show that

$$\frac{T_1}{25}=\frac{T_2}{25}=\frac{T_3}{14}=\frac{9W}{8\sqrt{4559}}.$$

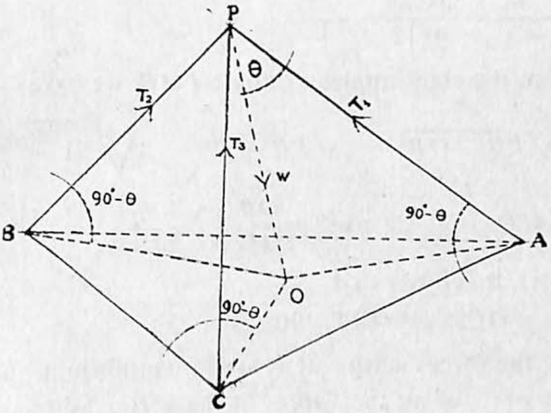


Fig. 38

We draw a vertical through *P* meeting the plane of the triangle *ABC* at *O*. We join *OA*, *OB* and *OC*. The thrusts T_1, T_2 and T_3 will act along *AP*, *PB* and *CP* respectively. The wright *W* will act at *P* vertically downwards along *PO*. We observe that

$\angle POA=\angle POB=\angle POC=90°$.

Also, $\triangle POB\cong\triangle POC\cong\triangle POA$ ......(1)

Hence, $OA=OB=OC$.

Hence, *O* is the circum-centre of $\triangle ABC$.

Let *R* be the radius of the circum-circle of $\triangle ABC$.

Let $BC=\text{a}$, $CA=\text{b}$ and $AB=\text{c}$.

$\therefore \text{a}=5$, $\text{b}=5$ and $\text{c}=6$.

$\therefore$The area $\triangle$ of the triangle *ABC* is given by

$$\triangle=\sqrt{S(S-\text{a})(S-\text{b})(S-\text{c})}$$

where $S=\frac{1}{2}(5+5+6)=8$

$$\therefore\ \triangle=\sqrt{8\times3\times3\times2}=12.$$

Now, from the properties of a triangle, we know that

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

$$\therefore \sin A=\frac{a}{2R}\ .$$

Also, $\triangle=\frac{1}{2}\text{bc}\sin A=\frac{1}{4}\ .\ \frac{\text{abc}}{R}$

$$\Rightarrow R=\frac{\text{abc}}{4\triangle}=\frac{5\text{x}5\text{x}6}{4\text{x}12}=\frac{25}{8}.$$

Now, from the right-angled triangle POB, we have

$$PO=\sqrt{PB^2-OB^2}=\sqrt{PB^2-R^2}=\sqrt{81-\frac{625}{64}}=\frac{\sqrt{4559.}}{8}$$

Let $\angle\ APO=\theta\ \therefore\ \cos\theta=\frac{OP}{AP}=\frac{\sqrt{4559.}}{72}$

$\therefore$From (1), it follows that

$$\angle OBP=\angle OCP=\angle OAP=90^\circ-\theta.$$

Since all the forces acting at P are in equilibrium, the algebraic sum of the resolved parts of all the forces in the vertical direction will be zero. This gives

$$W=(T_1+T_2+T_3)\sin(90^\circ-\theta)$$

$$=(T_1+T_2+T_3)\cos\theta=\frac{(T_1+T_2+T_3)\sqrt{4559}}{72} \qquad \ldots(2)$$

The horizontal components of the forces are $T_1\sin\theta$ along AO, $T_2\sin\theta$ along BO and $T_3\sin\theta$ along CO. These three components meeting at O and lying in the plane of $\triangle\ ABC$ must also be in equilibrium. Hence, by applying Lami's theorem, we get

$$\frac{T_1\sin\theta}{\sin\angle\ BOC}=\frac{T_2\sin\theta}{\sin\angle\ AOC}=\frac{T_3\sin\theta}{\sin\angle\ AOB}$$

$$\Rightarrow\ \frac{T_1}{\sin 2A}=\frac{T_2}{\sin 2B}=\frac{T_3}{\sin 2C}$$

Now, $A+B+C=\pi$

$\Rightarrow 2A=\pi-C$, since $A=B$

$$\therefore\ \frac{T_1}{\sin C}=\frac{T_2}{\sin C}=\frac{T_3}{2\sin C\cos C}$$

$$\Rightarrow T_1=T_2=\frac{T_3}{2\cos C}.$$

Now, $\cos C = \dfrac{a^2 + b^2 - c^2}{2\,ab}$

$= \dfrac{25 + 25 - 36}{2 \times 5 \times 5} = \dfrac{7}{25}$

Hence, $T_1 = T_2 = \dfrac{T_3}{\frac{14}{25}}$

$$\Rightarrow \frac{T_1}{25} = \frac{T_2}{25} = \frac{T_3}{14} = \frac{T_1 + T_2 + T_3}{64}$$

$$= \frac{72\,W}{64\sqrt{4559}} \quad \left[\text{from } (2)\right]$$

$$= \frac{9\,W}{8\sqrt{4559}}\,.$$

EXERCISE 3.1

1. Can a particle be ever kept in equilibrium by three forces of magnitudes P, Q and R if $P : Q : R$ is equal to (i) $3 : 2 : 1$ (ii) $10 : 4 : 5$ (iii) $\sqrt{2} : \sqrt{3} : \sqrt{5}$ or (iv) $4\sqrt{2} : 3\sqrt{5} : 7\sqrt{3}$?

   [Ans. (i) Yes (ii) No (iii) Yes (iv) Yes]

2. Forces of magnitudes 24*N*, 7*N* and 25*N* are in equilibrium. Show that two of them are at right angles.

3. Four forces of magnitudes 4*N*, 6*N*, x *N* and y *N* act at a point making angles of 30°, 60°, 120° and 240° with a fixed line respectively. If the forces are in equilibrium, find the values of x and y.

   $\left[\text{Ans. } x = \dfrac{4\sqrt{3}}{3},\ y = \dfrac{18 + 8\sqrt{3}}{3}\right]$

4. Three forces of magnitudes 8 *N*, 5 *N* and 4 *N* acting at a point are in equilibrium. Show that the angle between the two smaller forces is $\cos^{-1}\left(\dfrac{23}{40}\right)$.

5. Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another and conversely.

6. Three forces acting at a point are in equilibrium. If they are proportional to 3,5,7, show that the angles between the forces are 60°, $\cos^{-1}\dfrac{13}{14}$ and $\cos^{-1}\left(-\dfrac{11}{14}\right)$.

7. A particle of weight 50 N is suspended by two strings of lengths 3m and 4m attached to two points at the same level, whose distance apart is 5 m. Find the tensions in the string.

[Ans: $40N$ and $30N$]

8. A weight of 20 N rests in equilibrium on a smooth plane inclined at an angle of 60° with the horizontal, being held by two light strings, one acting upwards along the plane and the tension of the other is of magnitude 1 N acting along a line inclined at an angle of 30° with the former tension. Find the tension of the other string and the reaction of the plane.

$$\left[\text{Ans. Tension} = \frac{19}{2}\sqrt{3}\,N,\right.$$

$$\left.\text{reaction} = \frac{19}{2}\,N\right]$$

9. Find a point within (i) a triangle, (ii) a quadrilateral such that the forces represented by the lines joining it to the angular points may be in equilibrium.

[Ans: (i) The point of intersection of the medians.
(ii) The mid-point of the line joining the middle points of any pair of opposite sides.]

10. In a simple crane, the jib is 18 m long and is connected to the post by a chain 14 m long, attached at a point 12 m above the foot of the post. If the jib cannot bear a pressure more than $10^4\ N$, find the greatest weight that can be suspended from the end of the jib.

$$\left[\text{Ans. } \frac{2}{3} \times 10^4\ N\right]$$

(A simple crane consists of an upright post known as the king post, a chain known as the tie-bar and a rod known as jib at the end of which the weight is suspended. The chain is connected with the top of the post and the top of the jib. One end of the jib is attached with the lower part of the upright post and on the other end, the weight is suspended.)

11. A body of weight 10 N is suspended by two strings, 21 cm and 72 cm long, their other ends being fastened to the extremities of a rod of length 75 cm. If the rod be so held that the body hangs immediately below its middle point, find the tensions of the strings.

$$\left[\text{Ans. } 2\frac{4}{5}\ N \text{ and } 9\frac{3}{5}\ N\right]$$

12. A string *ABC* has its extremities tied to two fixed points *A* and *B* in the same horizontal line. *A* weight W is knotted at a given point C in the string. Prove that the tension in the portion CA is $\frac{W\mathrm{b}}{4\mathrm{c}\triangle}(\mathrm{c}^2+\mathrm{a}^2-\mathrm{b}^2)$, where a, b, c are the sides and $\triangle$ the area of the triangle *ABC*.

13. A particle of weight W may be supported on a smooth inclined plane by a force P acting horizontally or by a force Q acting parallel to the plane along the line of greatest slope. Prove that $\frac{1}{W^2}=\frac{1}{Q^2}-\frac{1}{P^2}$.

    If R and S be the pressures of the body on the plane in the two cases respectively, show that $RS=W^2$.

14. A particle weighing 10 *N* is supported by means of two strings attached to it. If the direction of one be at 60° to the horizontal, find the direction of the other in order that its tension may be as small as possible and the values of the tensions in the two strings in this case.

    [Ans. At right angles to the first string; $5\sqrt{3}$ *N* and 5 *N*]

15. A weightless wire is stretched between two points *A* and *B* on the same level 4m apart. A weight of 10 *N* is hung at the middle point of the wire and causes it to drop $\frac{1}{6}$ m below the line *AB*. Find the tension in the wire.

    [Ans. 60.2 *N*]

16. A smooth ring of weight W can slide freely along a string which has its ends attached to two fixed points. The ring is pulled horizontally with a force P. If the portions of the string are inclined at angles θ and ϕ to the vertical, prove that $P=W\tan\frac{1}{2}(\theta-\phi)$.

17. A weight of 30 *N* is supported by a string fastened to a point on a smooth plane inclined at an angle 15° to the horizon and the string is only just strong enough to support a weight of 15 *N*. The inclination of the plane to the horizon is now gradually increased. Show that the string will break when the inclination of the plane is 30°.

18. A string of length 2m is attached to two points *A* and *B* at the same level and at a distance of 1 m apart. A ring of weight 10 *N*,

slung on to the string, is acted on by a horizontal force P which holds it in equilibrium vertically below B. Find the tension in the string and the magnitude of P.

$$\left[\text{Ans. } 6\frac{1}{4}N,\ \mathrm{P} = 5\ N\right]$$

19. The side AB of an equilateral triangle ABC formed by three weightless inextensible strings A and B are tied to two fixed points D and E by equal weightless strings AD and BE. A weight of $3N$ is attached to C. If $\angle DAB = \angle ABE = 150°$, show that the tensions in the strings AB, AD, BE, AC and BC are respectively $\sqrt{3}\ N$, $3N$, $3N$, $\sqrt{3}\ N$ and $\sqrt{3}\ N$.

20. The planes AB and AC having a common height are inclined to the horizon at angles α and β respectively. Two weights, one in each plane, are kept in equilibrium by a string attached to the weights and passing over A. Show that the weights are in the ratio $AB : AC$.

21. A particle of weight W rests against a circumference of a circular plate whose plane is vertical. A cord attached to the particle passes over a pulley placed vertically above the highest point of the circle at a distance from the circle equal to the radius and it carries a weight P. Show that the particle will rest at an angular distance θ from the highest point of the circle such that $\cos\theta = \frac{5}{4} - \frac{P^2}{W^2}$.

22. A uniform plane lamina in the form of a rhombus, one of whose angles is 120°, is supported by two forces applied at the centre in the directions of the diagonals so that one side of the rhombus is horizontal. Show that if P and Q are the forces and $P > Q$, then $P^2 = 3Q^2$.

23. A small ring is placed at the centre of a regular hexagon and is kept in position by six strings drawn tight, all in the plane of the figure, and each fastened at the other end to an angular point of the hexagon,. If the tensions in four consecutive strings are $2N$, $7N$, $9N$ and $6N$ respectively, show that the tensions in the remaining strings are $3N$ and $13N$.

24. Three light equal strings are knotted together to form an equilateral triangle ABC and a weight W is suspended from A. If the funicular triangle and the weight be supported with BC horizontal by means of two strings of negligible weight at B and C, each

inclined at an angle of 135° to BC, show that the tension in BC is

$$\frac{W}{6}(3 - \sqrt{3}).$$

25. A weight of $10N$ hangs by a string from a fixed point. The string is drawn out of the vertical by applying a force of $5N$ to the weight. In what direction must this force be applied in order that, in equilibrium, the deflection of the string from the vertical may have its greatest value? What is the amount of the greatest deflection?

[Banaras, '41]

[Ans. At right angles to the string; 30°.]

26. A and B are two fixed points in the same level at a distance c apart. Two weightless cords AC and BC of lengths b and a respectively support a weight at C. Show that the tensions of the cords are in the ratio $b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$.

27. Two beads of weights W_1 and W_2 slide on a smooth circular wire in a vertical plane, being connected by a light thread which subtends an angle 2θ at the centre of the circle. When the beads are in equilibrium on the upper half of the wire, prove that the inclination ϕ of the string to the horizontal is given by

$$\tan\phi = \frac{W_1 - W_2}{W_1 + W_2}\tan\theta.$$

28. A particle is attracted by three centres of forces situated at the angular points of a triangle, the force of attraction being in each case proportional to the distance from the corresponding centre of force. Show that the position of rest of the particle is at the centroid of the triangle.

29. Three forces P, Q and R acting along OA, OB and OC are in equilibrium. If O be the circum-centre of the triangle ABC, prove that

(i) $$\frac{P}{\dfrac{1}{b^2}+\dfrac{1}{c^2}-\dfrac{a^2}{b^2c^2}} = \frac{Q}{\dfrac{1}{c^2}+\dfrac{1}{a^2}-\dfrac{b^2}{c^2a^2}} = \frac{R}{\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{c^2}{a^2b^2}}$$ [C.U. '38]

(ii) $$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}$$

30. O is the circum-centre of the triangle ABC and L, M and N are the feet of the perpendiculars from A, B and C respectively on the opposite sides. If the forces acting along OA, OB and OC

are in equilibrium, show that they are proportional to the sides of the triangle *LMN*.

31. A light string is fastened to two points *A* and *D* at the same level, the length of the string exceeding the distance *AD*, and particles of weights 2*N* and 1*N* are fastened to it at two points *B* and *C* respectively. If *AB*, *BC* and *CD* make angles α, β and γ respectively with the horizontal, prove that
$\tan\alpha = 2\tan\gamma \pm 3\tan\beta$.

32. (a) A light string passes over three smooth pegs *A*, *B* and *C* which are in the same horizontal line. From the extremities of the string are suspended two weights *P* and *P′*. Two other weights *W* and *W′* are knotted at the two points *D* and *E* of the string, where *D* lies between *A* and *B* and *E* lies between *B* and *C*.
If $AB=a$, $BC=b$, $DB+BE=l$, $\angle BAD=\theta$, $\angle ABD=\phi$, $\angle EBC=\phi'$ and $\angle ECB=\theta'$ and the system is in equilibrium, prove that

(i) $\dfrac{P}{W}=\dfrac{\text{cas}\ \phi}{\sin(\theta+\phi)}$

(ii) $\dfrac{P'}{W'}=\dfrac{\cos\phi'}{\sin(\theta'+\phi')}$

(iii) $\dfrac{W\cos\theta}{\sin(\theta+\phi)}=\dfrac{W'\cos\theta'}{\sin(\theta'+\phi')}$ and

(iv) $\dfrac{a\sin\theta}{\sin(\theta+\phi)}+\dfrac{b\sin\theta'}{\sin(\theta+\phi')}=l.$

(b) If in the above problem, the weights *W* and *W′*, instead of being knotted to two given points in the string, are attached to two smooth rings which are capable of sliding freely along the string, then prove that $\sin\theta=\sin\phi=\dfrac{W}{2P}$ and $\sin\theta'=\sin\phi'=\dfrac{W'}{2P'}$.

33. (a) One end of a string is attached to a fixed point *A*. The string after passing over a smooth peg *B*, sustains a given weight *P* at its other extremity. A particle of weight *W* is knotted at a given point *C* in the string.
If $AB=a$, $AC=b$, $\angle CAB=\theta$, $\angle CBA=\phi$ and the system is in equilibrium, prove that

$$\frac{P}{W}=\frac{\cos\theta\,(a^2-2ab\cos\theta+b^2)^{1/2}}{a\sin\theta}$$

(b) If in the above problem, the extremities *A* and *B* are fixed and the weight W is that of a smooth and heavy ring C, which is capable of sliding freely along the string, show that the horizontal force which must be applied to the ring C in order that $\angle CAB$ and $\angle CBA$ become equal to γ and δ respectively for equilibrium is $W \tan \dfrac{\gamma-\delta}{2}$.

34. Four pegs at A, B, C and D are lying in a vertical plane forming a square $ABCD$, the horizontal line AB being above CD. A string attached to A and B passes through a heavy smooth ring R of weight W, while another string is attached to D and R. The ring is kept in equilibrium half way between H, the middle point of DA and O, the centre of the square. Show that the tensions in the strings ARB and DR are

$$W.\frac{13\sqrt{5}+5\sqrt{13}}{32} \text{ and } W.\frac{\sqrt{5}+5\sqrt{13}}{16}$$

respectively.

35. A light string $DABCE$ passes over three smooth pegs A, B and C in a vertical plane and sustains two equal weights each equal to W from its extremities D and E such that $\angle DAB=\alpha$, $\angle ABC=\beta$ and $\angle BCE=\gamma$. Find the magnitudes of the angles α, β and γ when the system of pegs is least likely to break, the pegs being all equally strong.

$$\left[\text{Ans. } \frac{2}{3}\pi\right]$$

36. A series of equal weights are knotted at different points of a string, the two extremities of which are tied to two fixed points. Prove that, in the equilibrium position, the tangents of the inclinations to the horozontal of the successive portions of the string are in *A.P.*

37. Forces X and Y act along the sides AB and AD respectively of a cyclic quadrilateral $ABCD$. If they are balanced by a force Z which acts along the diagonal CA from C to A, show that $X:Y:Z=CD:CB:BD$.

38. Coplanar forces whose magnitudes are proportional to the sides of a closed polygon act perpendicularly to those sides at their middle points, all inwards or all outwards, prove that they are in equilibrium.

39. A heavy triangular lamina ABC is hung from a point O by three strings attached to the vertices A, B and C. If T_1, T_2 and

T_3 be the tensions of the strings OA, OB and OC, show that

$$\frac{T_1}{OA}-\frac{T_2}{OB}=\frac{T_3}{OC}.$$

40. A transversal cuts the lines of action of three forces P, Q and R which act at the point O and are in equilibrium at the points A, B and C. Show that with a convention regarding sign

$$\frac{P}{OA.\ BC}=\frac{Q}{OB.\ CA}=\frac{R}{OC.\ AB}.$$

41. A small bead can slide on a smooth elliptic wire being acted on by forces towards the foci, which are proportional to the corresponding focal distances. Prove that the only positions of equilibrium are the extremities of the axes.

42. A praticle is placed on a square table at distances C_1, C_2, C_3 and C_4 from the corners and to it are attached strings over smooth pulleys at the corners, and supporting weights P_1, P_2, P_3 and P_4. Prove that, if there is equilibrium,

$$\frac{\dfrac{P_1}{C_1}-\dfrac{P_3}{C_3}}{C_1^2-C_3^2}=\frac{\dfrac{P_2}{C_2}-\dfrac{P_4}{C_4}}{C_2^2-C_4^2}.$$

Chapter IV

Parallel Forces

4.1 Cases of failure of Parallelogram law

We have seen in the preceding chapters that if two or more forces act on a single particle, we can always find their resultant either by the parallelogram law or by resolving the forces along two mutually perpendicular directions. In case two forces act at two distinct points of a rigid body, they may be taken to act on a particle placed at a point where the lines of action of the forces interesct provided these lines are not parallel. Then their resultant can be found by the parallelogram law if the point of intersection lies outside the body, the magnitude and the line of action can still be found out if the point is rigidly connected with the body. But in the case of two or more parallel forces acting on a body, we can not find out the resultant by the parallelogram law of forces, because the lines of action of these forces are parallel and so they do not meet at a point. We shall, therefore, discuss how the resultant of parallel forces can be found out.

4.2 Like and Unlike parallel forces

Parallel forces are said to be *like* when they act in the same direction and *unlike* when they act in the opposite directions.

4.3 Resultant of two like parallel forces

Let P and Q be the magnitudes of two like parallel forces acting at points A and B respectively of a rigid body and let them be represented by $\overrightarrow{AK}$ and $\overrightarrow{BK}$ respectively. At A and B, we introduce two equal and opposite forces, each of magnitude F and let them be represented by $\overrightarrow{AC}$ and $\overrightarrow{BC}$ respectivley. The resultant of the like parallel forces P and Q will

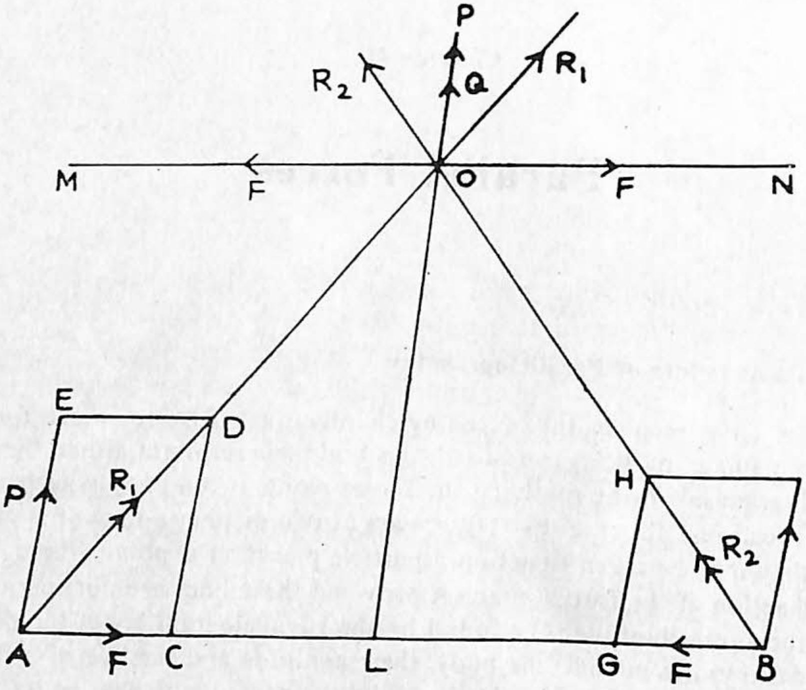


Fig. 39

not be affected due to the introduction of these two equal and opposite forces, since they balance each other. The parallelograms *ACDE* and *BGHK* are completed and the diagonals *AD* and *BH* are produced to meet at *O*. Through *O*, a line *OL* is drawn parallel to *AE* or *BK* to meet *AB* at *L*. Another line *MON* is drawn through *O* parallel to *AB*.

The forces *P* and *F* acting at *A* have a resultant R_1 represented by $\overrightarrow{AD}$ which may be supposed to act at *O*. Similary, the forces *Q* and *F* acting at *B* have a resultant R_2 represented by $\overrightarrow{BH}$ which may also be supposed to act at *O*. Now, the force R_1 at *O* can be resolved into two components: *F* along *ON* and *P* along *LO*. Similarly, R_2 at *O* can be resolved into the components *F* along *OM* and *Q* along *LO*. The two forces *F* acting at *O* along *ON* and *OM* are in equilibrium. Hence, the original like parallel forces *P* and *Q* are equivalent to a force of magnitude $(P+Q)$ acting along *LO*.

Poistion of the point L:

From similar $\triangle^s$ *ACD*, and *ALO* we have

$$\frac{AL}{LO}=\frac{AC}{CD}=\frac{F}{P}$$

$$\Rightarrow P.AL=F.LO \tag{i}$$

Similarly, from similar $\triangle^s$ BGH and BLO we have

$$\frac{BL}{LO}=\frac{BG}{GH}=\frac{F}{Q}$$

$$\Rightarrow Q.BL=F.LO \tag{ii}$$

From (i) and (ii), we have

$$P.\ AL=Q.\ BL$$

$$\Rightarrow \frac{AL}{BL}=\frac{Q}{P} \qquad \ldots\text{(iii)}$$

which shows that the point L divides AB internally in the inverse ratio of the given forces.

We have thus established the following theorem :

Theorem: The resultant of two like parallel forces of magnitudes P and Q acting at two different points A and B of a rigid body is a force of magnitude P+Q parallel to the lines of action of the forces P and Q, and act at a point on AB. which divides AB, internally in the inverse ratio of the forces P and Q.

Remark : (a) From (iii) : we have

$$\frac{P}{BL}=\frac{Q}{AL}=\frac{P+Q}{BL+AL}=\frac{R}{AB},\text{ where}$$

R is the resultant of the like parallel forces P and Q. Hence, a force R at L is equivalent to two parallel components P at A and Q at B on opposite sides of it provided $\frac{P}{BL}=\frac{Q}{AL}=\frac{R}{AB}$ and A, B, L are collinear.

(b) we observe from (iii) that if $P>Q$, then $AL<BL$ so that the resultant of two like parallel forces will pass through a point which is nearer to the greater force. If $P=Q$, then their resultant passes through the middle point of the line joining their points of application.

4.4 Resultant of two unequal and unlike parallel forces

Let P and Q be the magnitudes of two unlike parallel forces acting at points A and B respectively of a rigid body and let $P>Q$. Let the two forces be represented by $\overrightarrow{AE}$ and $\overrightarrow{BK}$ respectively. At A and B, we introduce two equal and opposite forces, each of magnitude F and let them

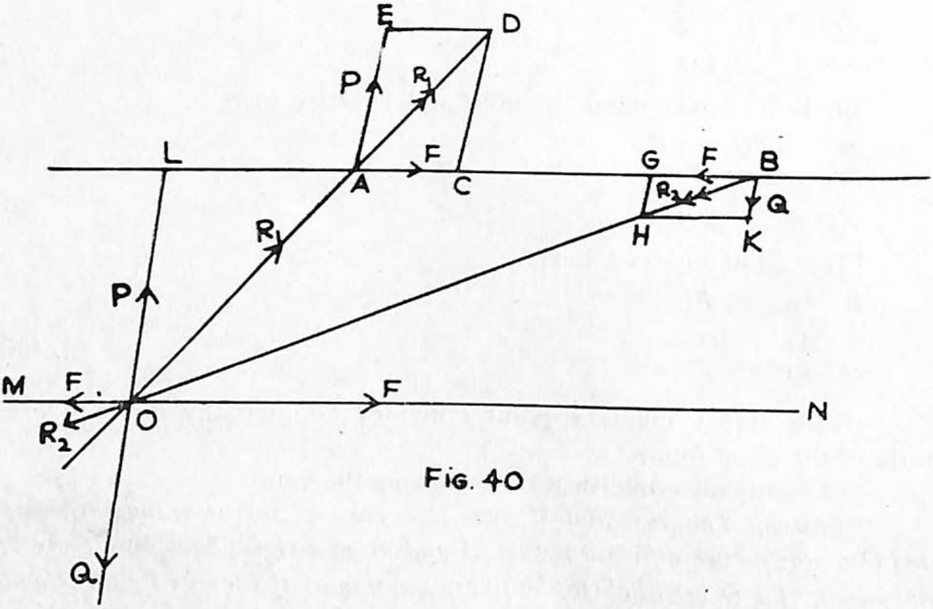


FIG. 40

be represented by $\overrightarrow{AC}$ and $\overrightarrow{BC}$ respectively. Since the body is rigid, these forces will be in equilibrium, and will not disturb the action of P and Q. The parallelcgrams $BGHK$ and $ACDE$ are completed and the diagonals BH and AD are produced to meet at O. (The diagonals will always meet unless they are parallel, which is the case when P and Q are equal). Through O, we draw a line OL parallel to AE or BK to meet AB produced at L. Another line MON is drawn through O parallel to AB.

The forces P and F acting at A have a resultant R_1 represented by $\overrightarrow{AD}$ which may be supposed to act O. Similarly, the forces Q and F acting at B have a resultant R_2 respresented by $\overrightarrow{BH}$ which maya lso be supposed to act at O. Now, the force R_1 at O can be resolved into two components : F along ON and P along OL. Similarly, R_2 at O can be resolved into the components F along OM and Q along LO. The two forces F acting at O along OM and ON are in equilibirum. Hence, the original unlike parallel forces P and Q are equivalent to a force of magnitude $(P-Q)$ acting along OL i.e. acting through L parallel to P in the same direction as P.

Position of the point L:

From similar triangles OAL and ACD, we have

$$\frac{AL}{LO}=\frac{AC}{CD}=\frac{F}{P}$$

$$\Rightarrow P.\,AL=F.LO \tag{i}$$

Similarly, from similar triangles OBL and BGH, we have

$$\frac{BL}{LO}=\frac{BG}{GH}=\frac{F}{Q}$$

$$\Rightarrow QBL=F.LO \tag{ii}$$

From (i) and (ii), we have

$$P.AL=Q.BL$$

$$\Rightarrow \frac{AL}{BL}=\frac{Q}{P} \quad \ldots\text{(iii)}$$

which shows that the point L divides AB externally in the inverse ratio of the forces. We have thus the following theorem :

The resultant of two unlike parallel forces, P and Q, of unequal magnitudes ($P>Q$) acting at two points A and B of a rigid body, is a force of magnitude $P-Q$ acting in the direction of the greater force P, and the resultant divides the line AB externally in the inverse ratio of the forces.

Remark 1: The resultant of two unlike parallel forces is nearer the greater force.

Remark 2: If $P=Q$, then $\triangle BGH\cong\triangle ACD$ and hence $\angle HBG=\angle DAC=\angle LAO$ so that $AD \parallel BH$. In such a case, we cannot get any resultant force. Such equal unlike parallel forces form a couple which will be discussed in the subsequent chapter.

4.5 Centre of parallel forces

(i) When all the forces are like

Let $P_1, P_2 \ldots P_n$ be n like parallel forces acting at points $A_1, A_2 \ldots A_n$ respectively of a rigid body. The resultant of the forces P_1 and P_2 acting at A_1 and A_2 is equal to P_1+P_2, and always passes through a point G_1 in A_1A_2 such that $P_1 . A_1G_1=P_2 . A_2G_1$ in whatever common direction the two forces act. Similarly, the resultant of two like parallel forces P_1+P_2 at G_1 and P_3 at A_3 is equal to $P_1+P_2+P_3$ and always passes through a fixed point G_2 in G_1A_3. In this way, the final resultant R of all the forces will be a like parallel force and

$$R=P_1+P_2+\ldots\ldots+P_n$$

which always passes through a point whose position is fixed relative to A_1, A_2, etc. Its position does not depend on the common direction of the like parallel forces.

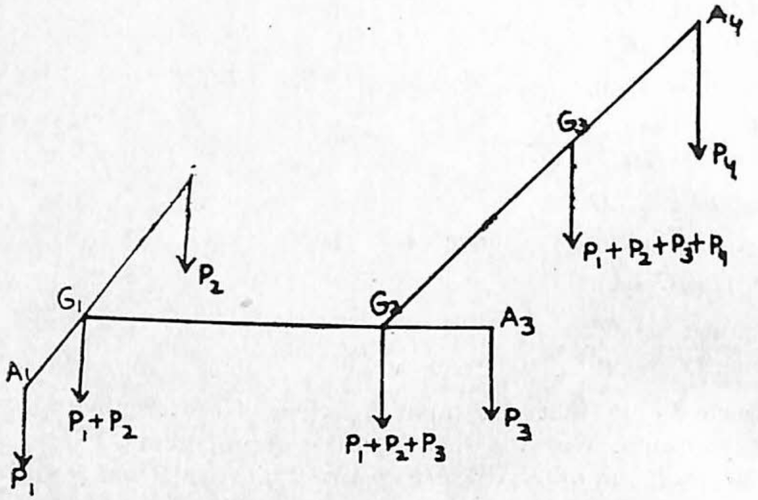


Fig. 41

This point is called the centre of parallel forces. Thus we can define the centre of parallel forces as follows: *The centre of parallel forces is a certain unique point through which the resultant of a given system of parallel forces, when it exists, passes and the position of this point depends on the magnitudes and the points of application of the forces and not on their directions.*

(*ii*) *When the forces are not all like*

In this case, we divide all the forces into two groups of like parallel forces. Let R_1 and R_2 be the resultants of the two groups of like parallel forces. Obviously, R_1 and R_2 will be two unlike parallel forces. If $R_1 \neq R_2$ and $R_1 > R_2$, then the required resultant of all the forces is $R_1 - R_2$ which is parallel to the system of forces and is equal to their algebriaic sum.

If $R_1 = R_2$ and their lines of action are coincident, then the system of forces will be in equilibrium. But if their lines of action are not coincident, then they will form a couple.

Remark: The above method of finding the resultant of a system of parallel forces is valid even if the forces are not coplanar.

4.6 Solved Examples

Ex. 1. The resultant of two unlike parallel forces of 10 N and 18 N acts along a line at a distance of 12 cm from the line of action of the smaller force. Find the distance between the lines of action of the two forces.

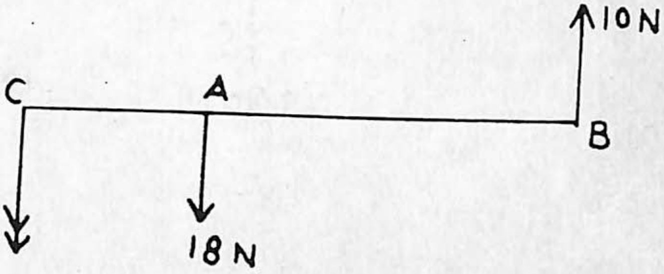


Fig. 42

Let the unlike forces 18 N and 10 N act at A and B respectively and let C be the point through which their resultant passes.

It is given that $BC=12$cm.

Now, $18\times AC=10\times BC=10\times 12$

$$\Rightarrow AC=\frac{120}{18}=6\frac{2}{3}$$

Hence, $AB=(12-6\frac{2}{3})$ cm$=5\frac{2}{3}$cm.

Ex. 2. Two men, one stronger than the other, have to remove a block of stone weighing 140 N with a light pole whose length is 6m. The weaker man connot carry more then 40 N of weight. Where must the stone be fastened to the pole so as just to allow him his full share of weight?

Let C be the position of the weight of 140 N at a distance of x from the weaker man at A carrying a weight of 40 N. The stronger man at B must, therefore, carry a weight of $(140-40)\ N=100\ N$. Since the resultant of like parallel forces 40 N and 100 N passes through C, we have

$$40\text{ x}=100(6-\text{x})$$

$$\Rightarrow 140\text{ x}=600$$

$$\Rightarrow \text{x}=\frac{600}{140}=4\frac{2}{7}.$$

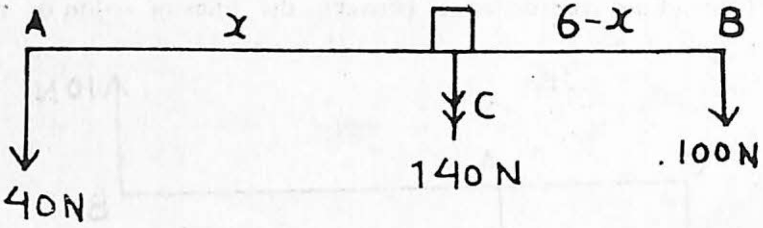


Fig. 43

Hence, the stone must be placed at a distance of $4\frac{2}{7}$ m from the weaker man.

Ex. 3. Two like parallel forces P and Q act at given points of a body. If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

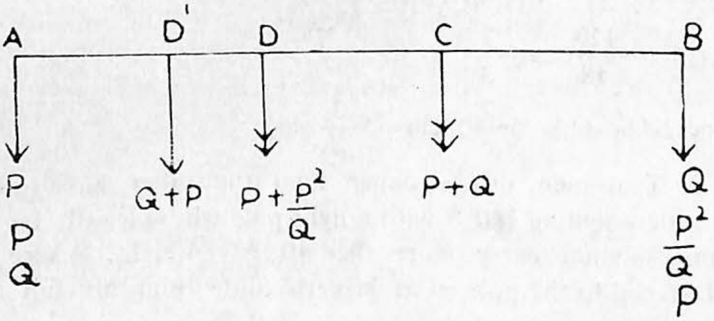


FiG 44

Let the resultant of two like parallel forces P at A and Q at B pass through C. When P and Q are interchanged, let their resultant pass through D' and when Q is replaced by $\frac{P^2}{Q}$, let the resultant of P at A and $\frac{P^2}{Q}$ at B pass through D. To show that D and D' are coincident.

We have

$Q.\ AD'=P.\ BD'$

$\Rightarrow Q.\ (AC-CD')=P.\ (BC+CD')$

$\Rightarrow (P+Q).\ CD'=Q.\ AC-P.\ BC$

$$\Rightarrow CD'=\frac{Q.\,AC-P.\,BC}{P+Q} \qquad \text{..(1)}$$

Again, $P.\,AD=\frac{P^2}{Q}.\,BD$

$$\Rightarrow P.\,(AC-CD)=\frac{P^2}{Q}.\,(BC+CD)$$

$$\Rightarrow CD\left(P+\frac{P^2}{Q}\right)=P.AC-\frac{P^2}{Q}.\,BC$$

$$\Rightarrow CD.\;\frac{P\,(Q+P)}{Q}=\frac{P\,(Q.\,AC-P.\,BC)}{Q}$$

$$\Rightarrow CD=\frac{Q.\,AC-P.\,BC}{P+Q}. \qquad \text{...(2)}$$

From (1) and (2) we see that

$CD=CD'$

i.e. D, and D' are coincident.

Hence, the lines of action of the resultants in the two cases will be the same.

Ex. 4. A force P acts along $A\,O$, where O is the circum-centre of the triangle ABC. Show that the parallel components of P acting at B and C are in the ratio sin $2B$: sin $2C$.

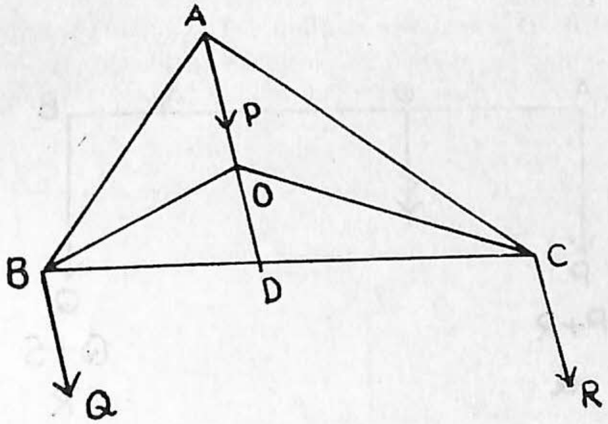


FIG. 45

Let Q and R be the parallel components of the force P.

Let AO produced intersect BC at D.

Since the resultant of the like parallel forces Q and R is P passing through B, we have

$Q.\ BD=R.\ DC$

$$\Rightarrow \frac{Q}{R}=\frac{DC}{BD}=\frac{\dfrac{DC}{OC}}{\dfrac{DB}{OB}}\quad\left[\because OA=OB=OC\right]$$

$$=\frac{\dfrac{\sin\angle DOC}{\sin\angle ODC}}{\dfrac{\sin\angle DOB}{\sin\angle ODB}}$$

$$=\frac{\sin\angle DOC}{\sin\angle DOB}\qquad\left[\begin{array}{l}\because \sin\angle ODC=\sin(180^\circ-\angle ODB)\\ \qquad\qquad\quad=\sin\angle ODB\end{array}\right]$$

$$=\frac{\sin(180^\circ-\angle AOC)}{\sin(180^\circ-\angle AOB)}=\frac{\sin\angle AOC}{\sin\angle AOB}=\frac{\sin 2B}{\sin 2C},$$

angle at the centre of a circle being double the angle at the circumference.

Hence, the result.

Ex. 5. The resultant of two like parallel forces P, Q passes through a point O. When P is increased by R and Q by S, the resultant still passes through O, and also when Q, R replace P, Q respectively. Show that

$$S=R-\frac{(Q-R)^2}{P-Q}.$$

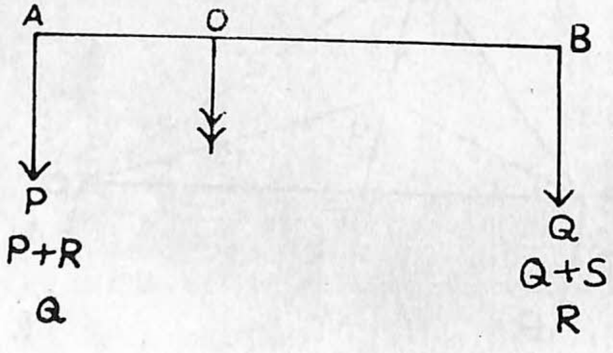


FIG. 46

Let the resultant of two like parallel forces P, Q or $P+R$, $Q+S$ or Q, R acting at A and B respectively pass through O, we then have

$$P.\ AO=Q.\ BO \qquad \ldots(1)$$

$$(P+R).\ AO=(Q+S).\ BO \qquad \ldots(2)$$

$$Q.\ AO=R.\ BO \qquad \ldots(3)$$

From (1) and (2) we have

$$R.\ AO=S.\ BO$$

$$\Rightarrow \frac{AO}{BO}=\frac{S}{R}.$$

Also, from (3),

$$\frac{AO}{BC}=\frac{Q}{R}$$

and from (1)

$$\frac{AO}{BO}=\frac{Q}{P}$$

$$\text{Hence,}\quad \frac{S}{R}=\frac{R}{Q}=\frac{Q}{P}=\frac{R-S}{Q-R}=\frac{Q-R}{P-Q}$$

$$\therefore \frac{R-S}{Q-R}=\frac{Q-R}{P-Q}$$

$$\Rightarrow R-S=\frac{(Q-R)^2}{P-Q}$$

$$\Rightarrow S=R-\frac{(Q-R)^2}{P-Q}.$$

Ex. 6. A uniform rod of length $2l$ and weight W is lying across two pegs on the same level c metre apart. If neither peg can stand a thrust T, find the greatest length of the rod which may project beyond either peg.

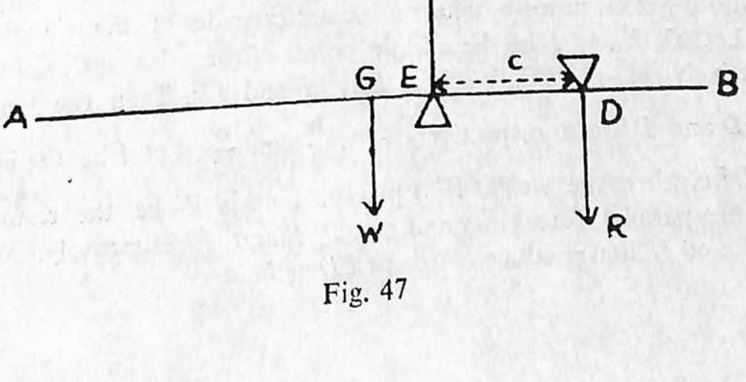


Fig. 47

Let AB be the rod and E, D be the positions of the two pegs, E being below the rod and D is above it. Since the rod is uniform, its weight will act at its middle point (see the chapter on Centre of Gravity).

Let the weight W act at G, the middle point of the rod. In order that the rod may be in equilibrium, the point G must be outside the pegs at E and D. Let R and S be the reactions of the pegs at D and E respectively so that R will act vertically downwards and S vertically upwards.

We have

$$S=W+R \qquad \ldots(1)$$

Since $S>R$, S will be the first to attain the maximum value T.

Let $AE=x$. $\therefore GE=x-l$

Now, $W.\ GE=R.\ ED$

$\Rightarrow W(x-l)=(S-W)\ c$ [from (1)]

$\Rightarrow S=W+\dfrac{W(x-l)}{c}$

Since $S\leqslant T$,

$\therefore W+\dfrac{W(x-l)}{c}\leqslant T$

$\Rightarrow cW-Wl+Wx\leqslant cT$

$\Rightarrow x\leqslant\dfrac{cT+lW-cW}{W}.$

$=l-c+\dfrac{cT}{W}$

Hence, the required greatest length of the rod which can project beyond the peg at E is $l-c+\dfrac{cT}{W}$.

Ex. 7. A weight W is placed upon a triangular table ABC at the incentre of the triangle formed by joining the mid-points of the sides of the table. Determine the pressure upon the legs and show that they are in the ratio $b+c : c+a : a+b$, where a,b,c are the sides of the triangle ABC.

Let D, E and F be the middle points of the sides BC, CA and AB respectively of $\triangle ABC$. We join DE, EF and FD. Then the lengths of EF, FD and DE are respectively $\dfrac{a}{2}$, $\dfrac{b}{2}$ and $\dfrac{c}{2}$. Let I be the in-centre of $\triangle DEF$, where the weight W is placed. Letthis W be the resultant of three like parallel forces x, y and z at D, E and F respectively. We join ID, IE and IF and produce EI to cut FD at M.

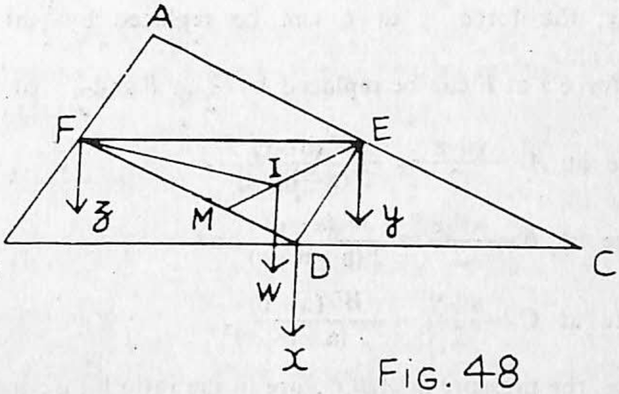


FIG. 48

Now, since the resultant of x, y and z passes through *I*, the resultant of x and z must pass through *M*.

$$\text{Hence, } \frac{z}{x} = \frac{MD}{FM} = \frac{\dfrac{MD}{IM}}{\dfrac{FM}{IM}} = \frac{\dfrac{\sin\angle MID}{\sin\angle MDI}}{\dfrac{\sin\angle MIF}{\sin\angle MFI}}$$

$$= \frac{\dfrac{\sin\left(\dfrac{E}{2}+\dfrac{D}{2}\right)}{\sin\dfrac{D}{2}}}{\dfrac{\sin\left(\dfrac{E}{2}+\dfrac{F}{2}\right)}{\sin\dfrac{F}{2}}} = \frac{\cos\dfrac{F}{2}}{\sin\dfrac{D}{2}} \times \frac{\sin\dfrac{F}{2}}{\cos\dfrac{D}{2}} \left[\because \frac{D}{2}+\frac{E}{2}+\frac{F}{2}=\frac{\pi}{2}\right]$$

$$= \frac{\sin F}{\sin D} = \frac{ED}{EF} = \frac{\frac{1}{2}c}{\frac{1}{2}a} = \frac{c}{a}$$

$$\therefore \frac{x}{a} = \frac{z}{c}.$$

Similarly $\frac{z}{c} = \frac{y}{b}$.

$$\text{Hence, } \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{x+y+z}{a+b+c} = \frac{W}{a+b+c}$$

$$\therefore x = \frac{aW}{a+b+c},\ y = \frac{bW}{a+b+c} \text{ and } z = \frac{cW}{a+c+b}.$$

Now, we replace the force x at *D* by two equal forces $\frac{x}{2}$ at *B* and $\frac{x}{2}$ at

C. Similarly, the force y at E can be replaced by $\frac{y}{2}$ at A and $\frac{y}{2}$ at C and the force z at F can be replaced by $\frac{z}{2}$ at B and $\frac{z}{2}$ at A. Hence,

the pressure at $A = \frac{y+z}{2} = \frac{W(b+c)}{2(a+b+c)}$;

the pressure at $B = \frac{x+z}{2} = \frac{W(a+c)}{2(a+b+c)}$ and

the pressure at $C = \frac{x+y}{2} = \frac{W(a+b)}{2(a+b+c)}$.

Hence, the pressure at A,B,C. are in the ratio $b+c : a+c : a+b$.

Ex. 8. If four parallel forces are in equilibrium and any plane meets their lines of action in four points, show that each force is proportional to the area of the triangle formed by the three points in which the other three forces meet the plane.

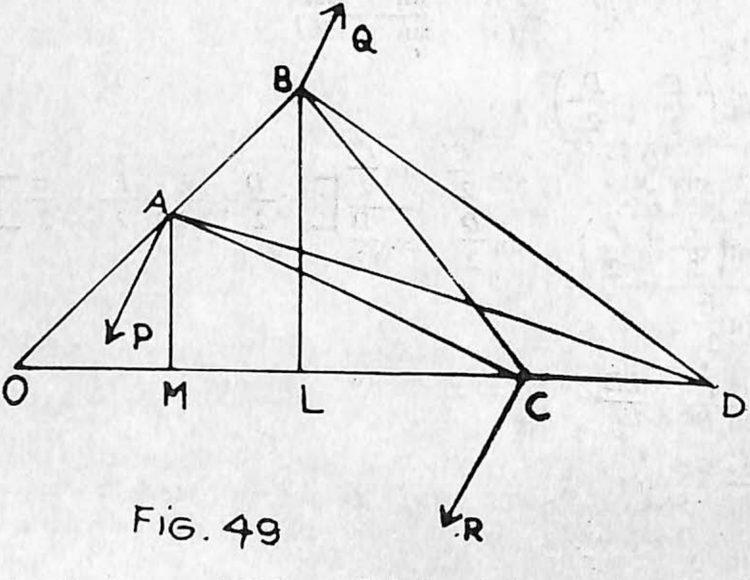


FIG. 49

Let the four parallel forces P,Q,R and S act at the points A,B,C and D respectively. We join AC, BC, AD, BD and produce BA and DC to meet at O. The resultant of P and Q act at some point on BA and that of S and R act at some point on DC. Since the forces are in equilibrium, these two resultants must pass through a common point of intersection of BA and DC i.e. through O. We have to prove that
$P : Q : R : S = \triangle BCD : \triangle ACD : \triangle ABD : \triangle ABC$.

Now, since the resultant of P and Q passes through O. hence

$P.\,OA = O.\,OB$

$$\Rightarrow \frac{P}{Q} = \frac{OB}{OA} = \frac{BL}{AM} = \frac{BL.\,CD}{AM.\,CD} = \frac{2\,\triangle BCD}{2\,\triangle ACD} = \frac{\triangle BCD.}{\triangle ACD}$$

Similarly, $\dfrac{Q}{R} = \dfrac{\triangle ACD}{\triangle ABD}$ and $\dfrac{R}{S} = \dfrac{\triangle ABD}{\triangle ABC}$.

Hence the result.

EXERCISE 4.1

1. Two men are carrying a straight uniform bar 16 m long weighing 80 N. One man supports it at a distance of 2 m from one end and the other man at a distance of 3m from the other end. What weight does each man bear?

[Ans. $43\frac{7}{11}\,N$ and $36\frac{4}{11}\,N$]

2. A horizontal light rod AB which is 4m long rests on two props at its extremities. A body of weight 60 N is suspended from a point C such that $AC = 1$m. Show that the pressure at A is three times that at B.

3. Find the magnitudes of two like parallel forces, acting at a distance of 4m apart which are equivalent to a force of 20 N acting at a distance of 1 m from one of the forces.

[Ans. 15 N and 5N, 15N force being 1m from the resultant.]

4. The extermities of a light pole rest on two smooth pegs A and B in the same horizontal line. A heavy load hangs from a point P of the pole. If $AP = 3\ PB$ and the pressure at B is 25 N more than that at A, find the weight of the load.

[Ans. 50 N]

5. A uniform see-saw plank, 6m long weighs 560 N. Find the position of the support when two children weighing 220 N and 340 N respectively sit at the two ends and the plank remains horizontal.

[Ans. $2\frac{19}{28}$ m from the heavier child].

6. If the position of the resultant of two parallel forces is unaltered when their positions are interchanged show that they are equal.

7. Two like parallel forces P and Q act at two points A and B of a rigid body, their resultant acting at C in the line AB. If Q is reversed ($Q > P$), the resultant acts at D in AB. If $CD = AC$, find P: Q.

[Ans. 1:3]

8. A man carries a bundle at the end of a stick which is placed horizontally over his shoulder. If the distance between his hand and his shoulder be changed, prove that the pressure on his shoulder varies inversely as the distance between his hand and shoulder.

9. A man carries a bundle at the end of a stick 2m long, which is placed on his shoulder. What should be the distance between his hand and shoulder in order that the pressure on the shoulder may be three times the weight of the bundle?

[Ans. $\frac{2}{3}$ m]

10. Two like parallel forces $\frac{P}{Q}$ N and $\frac{Q}{P}$ N have a resultant 2 N. Are these two forces equal? If R be the resultant of these forces and R' their resultant when they are unlike and If R: $R'=25:7$ and $P>Q$, find the ratio $P:Q$.

[Ans. Yes. 4:3].

11. Three like parallel forces P, $2P$ and $3P$ act at the angular points A,B and C respectively of a triangle ABC. Show that the resultant acts at a point G in AD where $AG:GD=5:1$ and D is a point on BC such that $BD:DC=3:2$.

12. Prove that the resultant of three equal like parallel forces passes through the centroid of the triangle if the forces act either at the angular points of the triangle or at the mid-points of the sides of the triangle.

13. Forces of magnitudes $2N$, $3N$, $4N$ and $5N$ act along the sides of a square, taken in order. Find their resultant and prove that it is parallel to a diagonal and find where it cuts the side along which the first force acts.

[Ans. $2\sqrt{2}\,N$ dividing the side externally in the ratio 7:9]

14. Forces of magnitudes P, $3P$, $2P$ and $5P$ act along the sides AB, BC, CD and DA of a square $ABCD$. Find the magnitude and direction of their resultant, and prove that it meets AD produced at a point L such that $AL:DL=5:4$.

[Ans. $\sqrt{5}\,P$ inclined an at angle $\tan^{-1}\frac{1}{2}$ with DA]

15. At the angular points of a square, taken in order, there act like parallel forces in the ratio 1:3:5:7. Show that the distance from the centre of the square of the point at which their resultant acts is $\frac{a}{4}$, where a is the length of the side of the square.

16. A,B,C and D are the angles of a parallelogram, taken in order. Like parallel forces proportional to 6, 10, 14 and 10 respectively act at the points A,B,C and D. Show that the centre and resultant of these

parallel forces remain the same, if, instead of these forces, parallel forces proportional to 8,12,16 and 4 act at the points of bisection of the sides AB, BC, CD and DA respectively.

17. Three like parallel forces P, Q and R act at the vertices A,B and C of the triangle ABC and are respectively proportional to a,b,c. Show that their resultant passes through the in-centre of the triangle.

18. Like parallel forces P, Q and R act at A,B and C, the vertices of a triangle ABC and O is any point within the triangle. Show that the resultant of P,Q and R will act at O if $P:Q:R=\triangle OBC:\triangle OCA:\triangle OAB$.

19. Like parallel forces of magnitudes P,P and $2P$ act at the vertices A,B and C respectively of a triangle ABC. Show that their resultant passes through the middle point of the line joining C to the middle point of AB.

20. A straight uniform heavy rod of length 3m has weights 50N and 100N attached to its ends and rests in equilibrium when placed across a fulcrum distant $\frac{1}{3}$m from the middle point. Fund the weight of the rod.

[Ans. 75N]

21. A unifrom rod 2m long and weighing 30N is laid on a table with $\frac{1}{2}$ m projecting over the edge. What weight can be hung on the end of the rod before it will be pulled over?

[Ans. 30 N]

22. Two light rods AB and BC of lengths in the ratio 1:2 are rigidly connected at B and meet at right angles. Weights of 8N and 4N are attached at A and C. Find the inclination of either rod to the horizon if the system can turn about B.

[Ans. 45°]

23. A light rod AB, 60 cm long, rests on two pegs whose distance apart is 25 cm. How must it be placed so that the reactions of the pegs may be equal when weights $2W$ and $3W$ respectively are suspended from A and B?

[Ans. The pegs are at distances 6.5 cm and 18.5 cm from the middle point of the rod.]

24. A uniform bar 2m long and weighing 34 N is suspended by two vertical strings. One is attached at a point 3 cm from one end and can just support a weight of 18 N without breaking. Another is attached 4 cm from the other end and can just support a weight of 20 N. A weight of $3\frac{2}{5}$ N is now attached to the rod. Show that the least distance of the

point of suspension of the weight without breaking either string is 7 cm form either string.

25. If two like parallel forces P and Q acting on a rigid body at A and B be interchanged in position, show that the point of application of the resultant will be displaced along AB through a distance d where $d=\frac{P-Q}{P+Q}AB,(P>Q)$.

[*C.U.* 54]

26. There are two like parallel forces P and Q. If two equal and unlike parallel forces S, S having their lines of action parallel to those of P and Q and distant c from one another be introduced anywhere in the plane, show that the resultant is displaced through a distance $\frac{cS}{P+Q}$.

27. Show that if each of two unlike parallel forces P, Q $(P>Q)$ acting at points A, B respectively be increased by S, their resultant will not be altered in magnitude, but its point of application will be moved through a distance

$$\frac{S.\ AB}{P-Q}.$$

28. P and Q are like parallel forces. If P is moved parallel to itself through a distance x, show that the resultant of P and Q moves through a distance

$$\frac{Px}{P+Q}.$$

29. The resultant of two parallel forces P and Q at A and B respectively acts at C when like, and at D when unlike. Prove that if parallel forces whose magnitudes are equal to these resultant forces, act simultaneously at C and D, then A and B will be the points at which their resultant will act in the two cases of like and unlike directions.

30. A line AB is divided into two parts at C. The resultant of two like parallel forces P and Q acting through the mid-points of AC and CB passes through C. If P and Q be interchanged in position, show that their resultant will pass through the mid-point of AB.

31. O is the ortho-centre of the triangle ABC. A force P acts along OA. This force is resolved into the like parallel forces acting at B and C. Show that the component at B is $P \sin B \cos C \operatorname{cosec} A$.

32. ABC is a triangle and O is any point within it. Like parallel forces act at A, B and C, which are proportional to the areas of triangles BOC, COA and AOB respectively. Show that the resultant acts at O.

33. Three like parallel forces P, Q and R act at the corners of a triangle ABC. If whatever be the common direction of the forces, the line of action of their resultant passes through

a) the centroid, show that $P=Q=R$

b) the in-centre, show that $\frac{P}{a}=\frac{Q}{b}=\frac{R}{c}$

c) the circum-centre, show that

$$\frac{P}{\sin 2A}=\frac{Q}{\sin 2B}=\frac{R}{\sin 2C}$$

d) the ortho-centre, show that

$$\frac{P}{\tan A}=\frac{Q}{\tan B}=\frac{R}{\tan C},$$

where a, b, c are the sides of the triangle.

34. A weight W is placed upon a triangular table ABC at the centre of the circumscribed circle. Show that the pressure on the leg A is $\frac{1}{2}$ $W \cos A \operatorname{cosec} B \operatorname{cosec} C$.

35. A weight W is placed upon a triangular table ABC at the intersection of the perpendiculars drawn from the angles to the sides. Show that the pressures at A, B and C are $W \cot B \cot C$, $W \cot C \cot A$ and $W \cot A \cot B$ respectively.

Chapter V

Moments

5.1 Introduction

We have seen that two or more concurrent forces acting at a point of a rigid body or a number of like or two unequal and unlike parallel forces acting at different points of a rigid body can always be combined to give a single resultant. This single resultant can produce a motion of translation of the body, if the body is free to undergo such a motion. When, however, a system of forces acts on a rigid body in different directions, then these forces may not meet at a point and so these forces may produce a motion of either translation or rotation or both. If a single force acts on a rigid body of which one point is fixed, then, unless the line of action of the force passes through the fixed point, it will tend to rotate the body about the fixed point. The case of rotation of a body introduces the idea of the *turning effect* or *moment of a force*, whose definition is given below.

5.2 Definition of Moment of a force about a point

The moment of a force about a point is the product of the force and the perpendicular distance of the point from the line of action of the force.

Thus the moment of a force P whose line of action is shown in fig 50 about a point O is Pp, where $p=ON$, the length of the perpendicular drawn from O to the line AB. It is clear that the moment of the force about the point is zero if either the line of action of the force passes through O i.e. if $p=o$ or the force P itself is zero.

5.3 Physical significance of a Moment

We observe that if the force P does not pass through a fixed point O of a body, then the product Pp (fig. 50) is a measure of the tendency of P to turn the body about O. The turning power is increased if either P is increased or its perpendicular distance ON from O is increased.

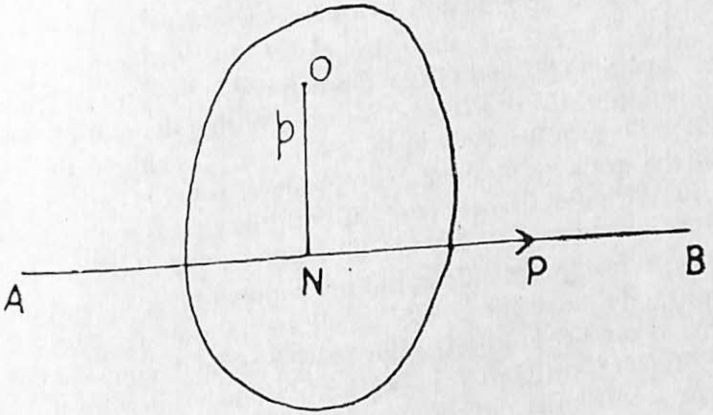


FIG. 50

Let us now consider a plane lamina resting on a smooth horizontal table. Let a point of the lamina be pinned at *O* about which the lamina can turn freely (fig. 51).

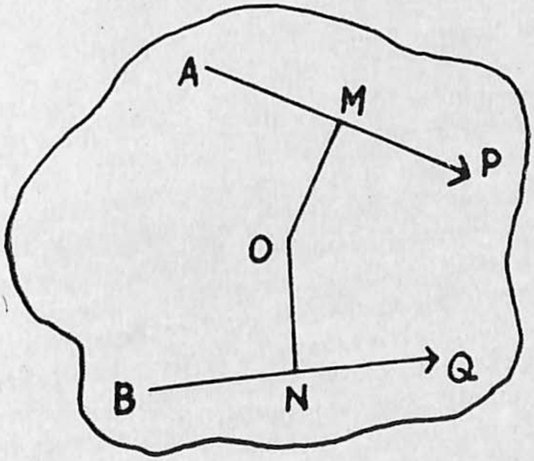


FIG. 51

Let the lamina be pulled by two strings whose extremities are fixed at A and B. Let P and Q be the tensions of the strings at A and B respectively and let OM and ON be the perpendiculars drawn from O on the lines of action of the two forces. We observe that the pull of the string at A alone will turn the body in the clockwise sense about the point O and that of the string at B alone will produce rotation of the lamina about O in the anticlockwise sense. If both the strings are pulled simultaneously, the direction of rotation of the lamina about O due to the joint effect of these two forces will depend not simply on the magnitudes of P and Q, but also on the distances OM and ON i.e. on the products $P \times OM$ and $Q \times ON$. It can be shown experimentally that the lamina will not turn at all if $| P \times OM | = | Q \times ON |$. On the other hand, it will rotate in the anticlockwise or clockwise direction due to the joint effect of the two forces according as $| Q \times ON | > | P \times OM |$ or $| P \times OM | > | Q \times ON |$.

We have thus seen that the magnitude of the tendency of rotation of a body about a fixed point due to a force depends on the moment of the force about the point and not on the magnitude of the force only.

Hence, the physical significance of the moment of a force about a point is that it is a fitting measure of the tendency of rotation of the body about the point caused by the application of the force.

5.4 Positive and Negative Moments

We have seen in the previous article that the force P turns the body in the clockwise sense and Q in the anticlockwise sense. Hence on account of the position of a force with respect to a fixed point, the moment of a force about a fixed point may be either in the anticlockwise or in the clockwise sense. Although either direction of rotation of the body may be chosen as positive, *the usual convention is that the moment is to be taken as positive if the force tends to rotate the body in the anticlockwise direction and it is to be taken as negative if the force tends to rotate the body in the clockwise direction.* When a number of forces act on a body the algebraic sum of the moments of all the forces about some fixed point in the body is obtained by giving the value of each moment its proper sign and adding them together.

5.5 Geometrical significance of a Moment

Let AB represent the given force P is magnitude, direction and line of action. Let O be any point outside AB and p, the length of the perpendicular ON from O on AB. We join OA and OB. Then the moment of

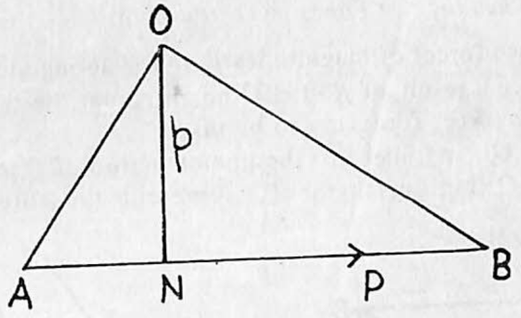


FIG. 52

the force P about O is $Pp = AB \times AN =$ twice the magnitude of the area of the triangle OAB.

Thus the geometrical significance of the magnitude of the moment of a force about a point is that it represents twice the area of the triangle formed by joining the point to the extremities of the line segment representing the force.

When proper sign is given to this expression, we get the moment completely in magnitude and sign.

5.6 Units of a Moment

The moment of a unit force about a point at a unit perpendicular distance from the line of action of the force is defined as the unit of the moment. In F.P.S. system, if the unit of force is poundal and the unit of distance be one foot, then the unit of moment is *foot-poundal*. In C.G.S. System, if the unit of force is dyne and the unit of distance be one centimetre, then the unit of moment is centimetre-dyne. In **S.I.** unit which we shall follow in this book, if the unit of force is 1 newton and the unit of distance is 1 metre then the unit of moment is newton-metre. In symbol this unit is written as Nm.

5.7 Moment of the resultant of two forces

Varignon's Theorem :

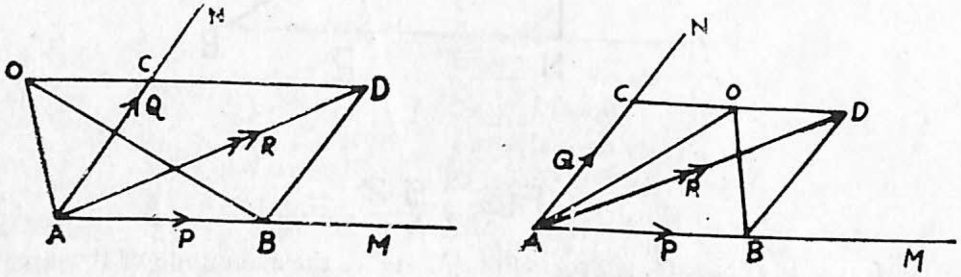
The algebraic sum of the moments of any two coplanar forces (except two equal unlike parallel forces) about any point in their plane is equal to the moment of their resultant about that point.

To prove this theorem, we consider two cases as follows:

Case I: When the two forces meet at a point

Let the two forces of magnitudes P and Q acting along AM and AN respectively have a resultant R. Let O be any point in their plane about which moments of the forces are to be taken.

We draw OC parallel to the line of action of P to meet the line of action of Q at C. Let the length AC represent the force Q and let the length AB represent the force P on the same scale. We now complete the parallelogram $ABCD$ and join the diagonal AD. Then AD represents the resultant R.



(i) (ii)

FIG. 53

Now, when the point O lies outside $\angle BAC$ [fig. 53 (i)], the moments of the forces P, Q and R about O will be positive according to this figure. Hence, the algebraic sum of the moments of P and Q about O

$=2\triangle AOB+2\triangle AOC$

$=2\triangle ABD+2\triangle AOC$ [$\because \triangle AOB$ and $\triangle ABD$ stand on the same base and between the same parallels.]

$=2\triangle ADC+2\triangle AOC$

$=2\triangle AOD$

$=$ moment of R about O.

When the point O lies inside $\angle BAC$ [fig. 53 (ii)], the moments of P and R about O will be positive and that of Q about O will be negative.

Hence, the algebraic sum of the moments of P and Q about O in this case

$=2\triangle AOB-2\triangle AOC$ [$\because$ the moment of P about O is anticlockwise

$=2\triangle ADB-2\triangle AOC$ and that of Q about O is clockwise]

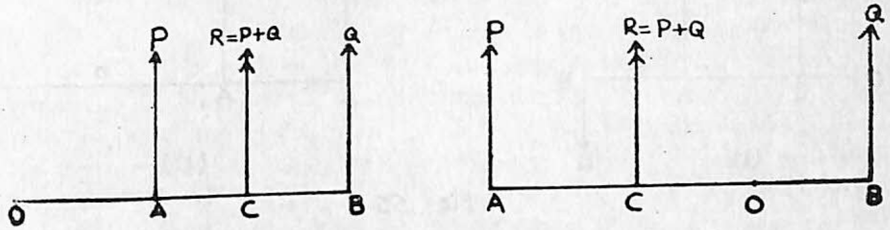
$=2\triangle ADC-2\triangle AOC$

$=2\triangle AOD$

$=$ moment of R about O.

Case II : *When the two forces are parallel*

(*a*) *Like parallel forces*



(i) (ii)

FIG. 54

Let the like parallel forces P and Q have a resultant $R=P+Q$. Let O be any point in their plane, about which the moments of the forces are to be taken. Through O, we draw a line perpendicular to the lines of action of the forces P, Q and R respectively to meet them at A, B and C respectively.

When the point O lies outside the lines of action of P and Q [fig. 54(i)], then the algebraic sum of the moments of P and Q about O

$$
\begin{aligned}
&= P.\ OA + Q.\ OB\\
&= P.\ (OC-AC)+Q.\ (OC+CB)\\
&= (P+Q).\ OC+Q.\ CB-P.\ AC\\
&= (P+Q).\ OC\ (\because P.\ AC=Q.\ CB)\\
&= R.\ OC\\
&= \text{moment of the resultant about } O.
\end{aligned}
$$

When the point O lies inside the lines of action of P and Q [fig. 54(ii)], then the algebraic sum of the moments of the forces P and Q about O

$$
\begin{aligned}
&= Q.\ OB-P.\ AO\\
&= Q.\ (CB-CO)-P.\ (AC+CO)\\
&= -(P+Q).\ CO+Q.\ CB-P.\ AC\\
&= -(P+Q).\ CO\ (\because P.\ AC=Q.\ CB)\\
&= -R.\ CO\\
&= \text{moment of the resultant about } O.
\end{aligned}
$$

(*b*) *Unlike parallel forces*

Let the unlike parallel forces P and Q ($P>Q$) have a resultant $R=P-Q$.

Let O be any point in their plane. Through O, we draw a line perpendicular to the lines of action of the forces P, Q and R to meet them at A, B and C respectively.

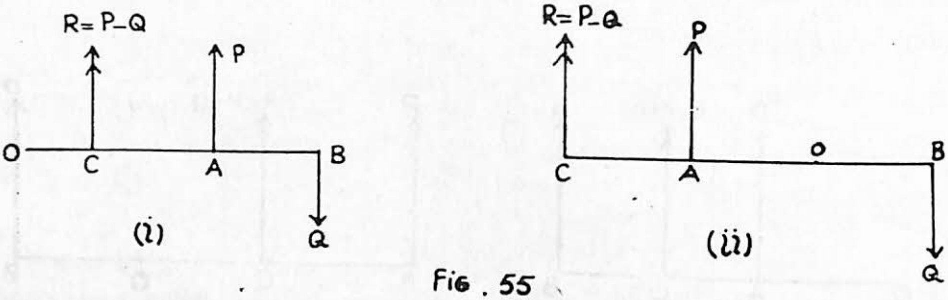


Fig. 55

When the point O lies outside the lines of action of P and Q [fig. 55(ii)], then the algebraic sum of the moments of P and Q about O

$=P.\ AO-Q.\ BO$
$=P.\ (CO+AC)-Q.\ (BC+CO)$
$=(P-Q).\ CO+P.\ AC-Q.\ BC$
$=R.\ CO\ (\because P.\ AC=Q.\ BC)$
$=$moment of the resultant about O.

When the point O lies inside the lines of action of P and Q [fig. 55(ii)], then the algebraic sum of the moments of P and Q about O

$=-Q.\ BO-P.\ AO$
$=-Q.\ (CB-CO)-P.\ (CO-AC)$
$=-(P-Q).\ CO-P.\ CA-Q.\ BC$
$=-R.\ CO\ \ [\because P.\ AC=Q.\ CB]$
$=$moment of the resultant about O.

Remark: If the point O lies on the line of action of the resultant, then the algebraic sum of the moments of the forces P and Q about the point is zero. Conversely, if the algebraic sum of the moments of two forces P and Q about any point O is zero, then either the resultant is zero in which case the forces are in equilibrium or the resultant passes through O.

5.8 Generalised theorem of Moments

Extension of Varignon's theorem :

If any number of coplanar forces acting on a body have a resultant, then the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

Let P_1, P_2, P_3, ... P_n be the forces and let R be their resultant. Let O be any point in their plane about which the moments are to be taken. Let R_1 be the resultant of the two forces P_1 and P_2. Then by section 5.7, the moment of R_1 about O is equal to the algebraic sum of the moments of P_1 and P_2 about O. We now combine R_1 with the third force P_3 and let R_2 be their resultant. Then the moment of R_2 about O=the algebraic sum of the moments of R_1 and P_3 about O=the algebraic sum of the moments P_1, P_2 and P_3 about O. Proceeding in this manner, we finally arrive at the final resultant R of the system and the moment of R about O=the algebraic sum of the moments of P_1, $P_2 \ldots P_n$ about the same point. If we denote the perpendicular distances from O on the lines of a action of the forces P_1, $P_2, \ldots P_n$ by $p_1, p_2, \ldots p_n$ respectively and if d be the perpendicular distance from O on the line of action of the final resultant R, then we have

$$\sum_{r=1}^{n} P_r p_r = Rd.$$

Remarks : If follows from above that if $\Sigma P_r\, p_r = O$, then either $R=0$ or $d=0$. When $R=0$, the system of forces is in equilibrium and when $d=0$, the resultant passes through O. Hence, *if the algebraic sum of the maments of any number of coplanar forces about any point in their plane be zero, then either the resultant passes through the point or the system of forces is in equilibrium.*

Conversely, if $R=0$, then $\Sigma P_r\, p_r=0$ i.e. *if a system of coplanar forces be in equilibrium, then the algebraic sum of their moments about any point in their plane is zero.*

If $d=0$ i.e. the point O lies on the line of action of the resultant of the system of forces, then also $\Sigma P_r\, p_r=0$. Hence, *the algebraic sum of the moments of any number of coplanar forces about any point on the line of action of their resultant is also zero.*

5.9 Moment of a force about an axis

We have so far considered the moments of forces about a point only in the case when all the forces together with the point are coplanar. In the general case, a system of forces acting on a rigid body may not be coplanar. In such cases, we may be required to measure the tendency of rotation of the rigid body due to these non-coplanar forces about some fixed line called the axis of rotation. This necessitates the definition of the moment of a force about a line when the line of action of the force and the given line are skew i.e. non-coplanar.

Let P be the force acting at any point O of a rigid body and AB is the line about which the moment is to be found out. We can consider the force P to lie on one of the faces of a rectangular parallelopiped and the line AB as one of its edges, which does not lie on this face as shown in the figure (fig. 56).

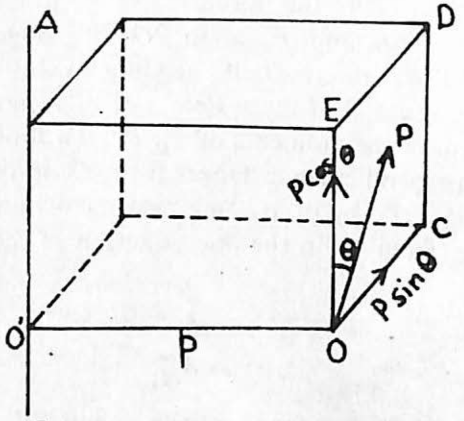


FIG. 56

Let $p = OO'$ be the shortest distance between the line of action of P and AB. Since $O'O$ is perpendicular to the face $EOCD$ on which the force P lies, the line OO' must be perpendicular to the line of action of the force P. Let the angle between the line of action of P and AB be θ. We resolve the force P along OE, an edge of the parallelopiped, parallel to AB and along OC, another edge. The component parallel to AB is $P \cos \theta$ and that along OC is $P \sin \theta$. It is clear that $P \cos \theta$ has no turning effect about AB, since OE is parallel AB and the moment of $P \sin \theta$ about AB

$= P \sin \theta . P$, [since $OC \perp AB$].

Hence, *the moment of a force about an axis is the product of the force in a direction perpendicular to the axis and the shortest distance between the axis and the line of action of the force.*

Remark 1 : The moment of a force about the line AB is zero if either the line of action of the force is parallel to AB or if the line of action of the force intersects AB in which case both the lines will be coplanar.

Remark 2 : The moment of a force about a point in its plane in two dimensions is, in fact, the particular case of the moment of the force about an axis through the point, perpendicular to the plane of the force.

Remark 3 : It can be shown that Varignon's theorem is true in the case of moments of a system of forces about a line also, i.e. *if a system of forces acting on a body has a resultant, then the algebraic sum of their moments about any line in the body is equal to that of their resultant.*

It follows that if a system of forces acting on a body is in equilibrium, then the algebraic sum of their moments about ony line in the body is zero.

5.10 Solved Examples

Ex. 1. Four parallel forces 3*N*, 6*N*, 9*N*and 12*N* acting at equal distances along a straight horizontal rod. Snow that their resultant will act at the same point when the third force is removed.

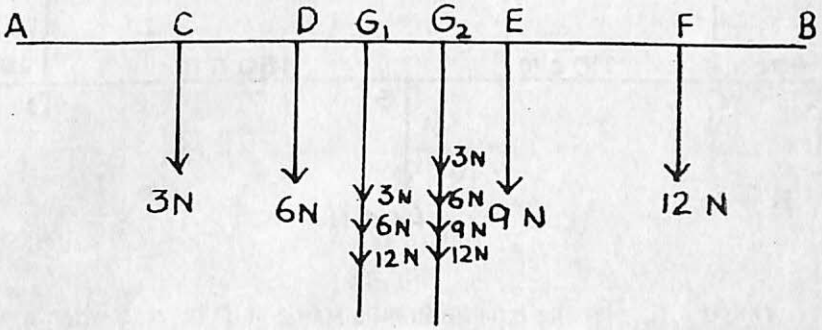


FIG. 57

Let the four sorces act at *C*, *D*, *E* and *F* respectively of the rod *AB*. Let *CD*=*DE*=*EF*=a and let the resultant of all these four forces pass through G_2 where DG_2=x. When the third force i.e. 9*N* force is removed from *E*, let the resultant of the remaining three forces passes through G_1 where DG_1=y. To show that x=y.

Since the resultant of the forces passes through G_2, the algebraic sum of their moments about G_2, will be zero. Hence, taking moments about G_2, we get

$$3(a+x)+6x-9(a-x)-12(2a-x)=0$$
$$\Rightarrow 30x=30a \Rightarrow x=a.$$

Again, since the resultant of the forces 3*N*, 6*N* and 12*N* acting at *C*, *D* and *F* respectively, passes through G_1, the algebraic sum of the moments of their resultant about G_1 will be zero. Hence, taking moment about G_1. we get

$$3(a+y)+6y-12(2a-y)=0$$
$$\Rightarrow 21y=21a \Rightarrow y=a.$$

Hence, x=y.

Hence the result.

Ex. 2. A uniform plank 3m long weighing 400 *N* is supported horizontally by two vertical strings, attached at distances 40 cm and 2m

30 cm from one end. What weight should be placed on the plank at one end so that (i) the tension in one of the strings just vanishes, (ii) the tension in one string is double the tension in the other.

Let AE be the plank. Its weight $400N$ acts at G, the middle point of the plank. The two strings are attached to the plank at C and D respectively at distances 40 cm and 20 cm respectively from the two ends.

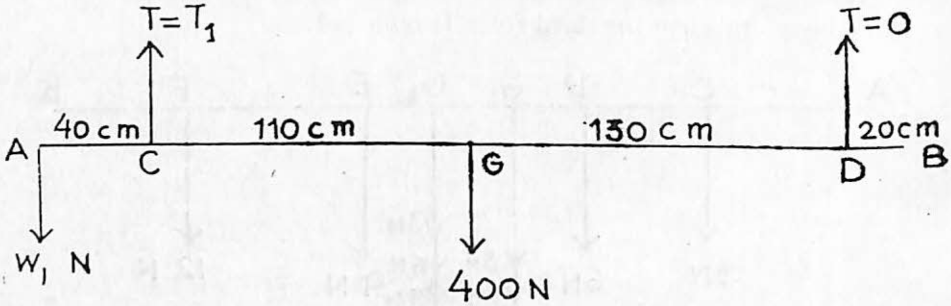


Fig. 58

In case (i), let the tension in the string at D be zero when a weight of W_1 is placed at the other end A of the plank and let T_1 be the tension of the string at C. Since the plank is in equilibrium under the forces W_1, T_1 and 400, all measured in newton, the algebraic sum of the moments of all these forces about any point will be zero. The most convenient point about which the moment is to be taken is the point C, since the moment of the unwanted force T_1 about C will be zero. Hence, taking moment about C, we get

$$W_1 \times 40 = 400 \times 110 \Rightarrow W_1 = 1100.$$

Hence, the weight that must be placed at A in case (i) is 1100 N.

In case (ii), let W_2 be the weight placed at A so that the tension of the nearest string at C becomes double the tension in the other string at D. Let these two tensions be $2T$ and T respectively. Since, the system of forces is in equilibrium in this case also, the algebraic sum of the moments of all the forces about any point will be zero. Taking moments about C, we get,

$$W_2 \times 40 - 400 \times 110 + T \times (130+110) = 0$$

$$\Rightarrow W_2 - 1100 + 6T = 0 \qquad \text{...(1)}$$

Also, for equilibrium, the resultant of the upward like parallel forces will balance the resultant of the downward like parallel forces. Hence,

$$T + 2T - 400 + W_2$$

$$\Rightarrow 3T = 400 + W_2$$

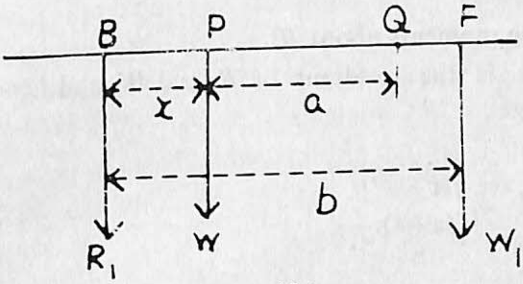
$$\Rightarrow 6T = 800 + 2W_2 \qquad \text{...(2)}$$

Eliminating T from (1) and (2), we get

$W_2-1100+800+2W_2=O\Rightarrow 3W_2=300\Rightarrow W_2=100.$

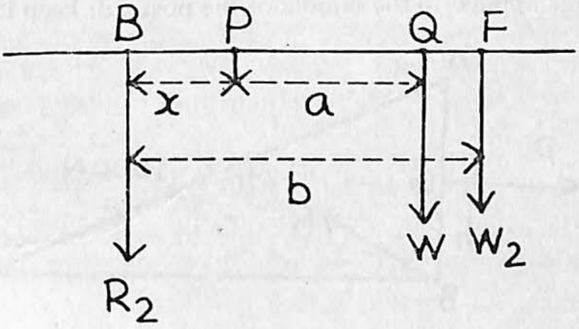
Hence, the weight that must be placed at A in case (ii) is $100N$.

Ex. 3. Prove that if a passenger of weight W advances a distance along the top of a motor bus, a weight $W\frac{a}{b}$ is transferred from the back springs to the front springs, where b is the distance between the axles.



(i)

FIG. 59



(ii)

FIG. 59

Let F and B be the front and back springs respectively so that FB=b. Let the passenger of weight W start from P and come to Q where PQ=a. When the passenger is at P, let the pressures at B and F be R_1 and W_1 respectively [fig. 59 (i)] and when the passenger comes to Q, let the pressures at these two points be R_2 and W_2 respectively [fig. 59 (ii)]. To prove that the pressure at F is increased by $\frac{W\text{a}}{\text{b}}$ when the passenger comes to Q from P i.e. to prove that $W_2 - W_1 = \frac{W\text{a}}{\text{b}}$.

Let $BP = \text{x}$ $\therefore QB = \text{a} + \text{x}$.

Since W is the resultant of R_1 and W_1 [fig. 59 (i)], we have, by Varignon's theorem,

$W\text{x} = W_1\,\text{b}$ [taking moments about B]. ...(1)

In fig. 59(ii), W is the resultant of R_2 and W_2 and hence taking moments about B, we get.

$W(\text{a}+\text{x}) = W_2\text{b}$. ...(2)

From (1) and (2), we get

$$W_1 = \frac{W\text{x}}{\text{b}} \text{ and } W_2 = \frac{W(\text{a}+\text{x}).}{\text{b}}.$$

$$\text{Hence, } W_2 - W_1 = \frac{W(\text{a}+\text{x}-\text{x})}{\text{b}} = \frac{W\text{a}}{\text{b}}.$$

Hence the result.

Ex. 4. A man can exert a force of 1000 N and pulls on a rope fastened to the top of a post, the rope being twice the length of the post. What horizontal force applied to the middle of the post will keep it from falling?

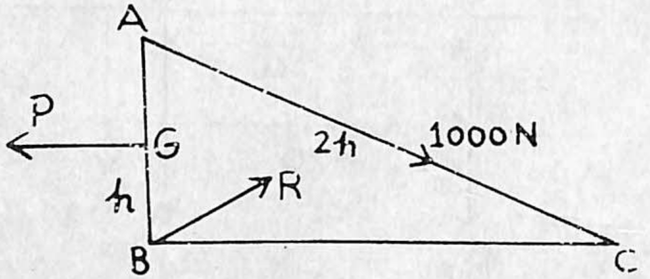


FIG. 60

Let AB be the post of height h, $A\,C$ the rope of length 2h and C be the position of the man. Let a horizontal force P act at G, the middle point of AB. Let the reaction of the ground on the post be R. Since these three forces are in equilibrium, hence the algebraic sum of the moments of these forces about any point in their plane will be zero.

Taking moments about B, we get

$$P\,\frac{h}{2}=1000\times BC\sin\angle BCA$$

$$=1000\times\sqrt{4h^2-h^2}\times\frac{h}{2h}$$

$$=1000\times\tfrac{1}{2}\times\sqrt{3}\,h$$

$$\Rightarrow P=1000\sqrt{3}.$$

Hence, the required horizontal force is $1000\sqrt{3}\ N$.

Ex. 6. Forces $\sqrt{2}, \sqrt{2}$ and $\sqrt{3}+1$ act along the sides BA, CA and BC of an isosceles triangle ABC whose vertical angle is 150°. Show that their resultant bisects BC and is parallel to one of the two equal forces.

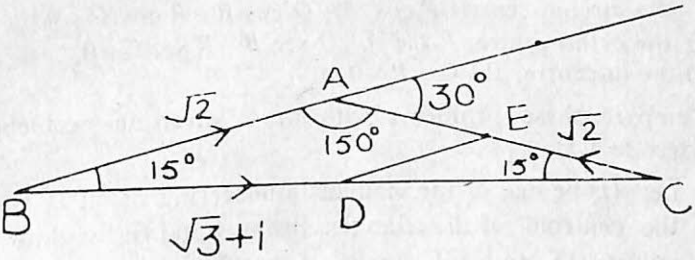


FIG. 61

Since ABC is an isosceles triangle and $\angle BAC=150°$, hence $\angle ABC=\angle ACB=15°$. Let the resultant intersects BC at D and AC at E. Hence, the algebraic sum of the moments of the forces about D and E will be separately zero.

Taking moments about D, we get

$$\sqrt{2}\,BD\sin 15°=\sqrt{2}\,DC\sin 15°$$

$$\Rightarrow BD=DC.$$

Hence, D is the middle point of BC i.e. the resultant bisects BC.

Let $AB=AC=a$ and $AE=x$ so that $EC=a-x$.

Now, taking moments about E, we get

$$(\sqrt{3}+1)\,EC\sin 15°-\sqrt{2}\times AE\sin 30°=0$$

$$\Rightarrow(\sqrt{3}+1)(a-x)\times\frac{(\sqrt{3}-1)}{2\sqrt{2}}-\sqrt{2}\times\frac{x}{2}=0$$

$$\left[\because \sin 15°=\sin(45°-30°)=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}\right]$$

$$\Rightarrow \frac{a-x}{\sqrt{2}} - \frac{x}{\sqrt{2}} = 0$$

$$\Rightarrow a-x-x=0 \Rightarrow x = \frac{a}{2}.$$

Hence, $AE=EC$ i.e. the resultant passes through the middle point E of AC. It follows that the line of action of the resultant of the three given forces is parallel to AB.

Ex. 6. Three forces P, Q and R act in the same sense along the sides BC, CA and AB of a triangle ABC. Show that if their resultant passes through

(i) the centroid, $P \operatorname{cosec} A + Q \operatorname{cosec} B + R \operatorname{cosec} C = 0$
(ii) the circum-centre, $P \cos A + Q \cos B + R \cos C = 0$
(iii) the ortho-centre, $P \sec A + Q \sec B + R \sec C = 0$
(iv) the in-centre, $P+Q+R=0$.

(Compare these prolbems with those given in problem no. 33, Exercise 4.1).

(i) Let AD be one of the medians intersecting BC at D and let G be the centroid of the triangle. From A and G, we draw perpendiculars AT and GL on BC. Let $AT=p$ [fig. 62(i)]. Then from similar triangles ATD and GLD, we have $GL=\frac{1}{3}p$.

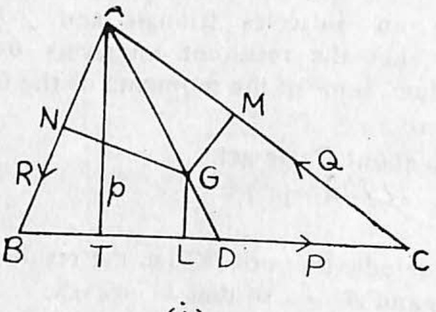


Fig. 62

Now, $\frac{1}{2}ap=\triangle$, where $\triangle$ is the area of $\triangle ABC$.

$$\text{Hence, } p = \frac{2\triangle}{a}$$

$$\therefore GL = \frac{2}{3} \cdot \frac{\triangle}{a}.$$

Similarly, if GM and GN be the perpendiculars from G on AC and AB respectively, then

$$GM=\frac{2}{3}\cdot\frac{\triangle}{b} \text{ and } GN=\frac{2}{3}\cdot\frac{\triangle}{c}.$$

Since the resultant of P, Q and R passes through G, the algebraic sum of the moments of all these forces about G will be zero. i.e.

$$P.\,GL+Q.GM+R.\,GN=0$$

$$\Rightarrow P\,\frac{2}{3}\cdot\frac{\triangle}{a}+Q\,\frac{2}{3}\cdot\frac{\triangle}{b}+R\,\frac{2}{3}\cdot\frac{\triangle}{c}=0$$

$$\Rightarrow \frac{P}{a}+\frac{Q}{b}+\frac{R}{c}=0$$

Now, since $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=K$ (say)

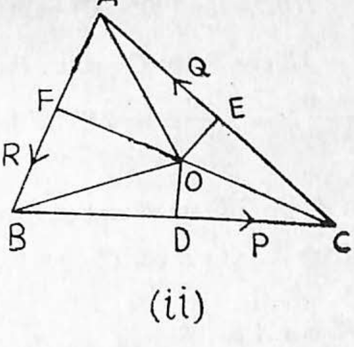
we have

$a=K\sin A$,
$b=K\sin B$,
and $c=K\sin C$.

Hence, $\dfrac{P}{\sin A}+\dfrac{Q}{\sin B}+\dfrac{R}{\sin C}=0$

$$\Rightarrow P\operatorname{cosec} A+Q\operatorname{cosec} B+R\operatorname{cosec} C=0.$$

(ii) Let O be the circum-centre of $\triangle ABC$. We join OA, OB and OC so that $OA=OB=OC=R'$, the circum-radius. Let D, E and F be the feet of the perpendiculars drawn from O on BC, CA and AB respectively. [fig. 62. (ii)]. Since the angle at the centre of a circle subtended by an arc is double the angle on the circumference subteneded by the same arc, hence $\angle BOC=2A$.



(ii)

Fig. 62

Let C be the centre of the wheel, through which the weight W is acting vertically downwards. A is the point of contact of the wheel with the ground. O is the obstacle of height h above the ground and $ON \perp CA$, where CA is vertical.

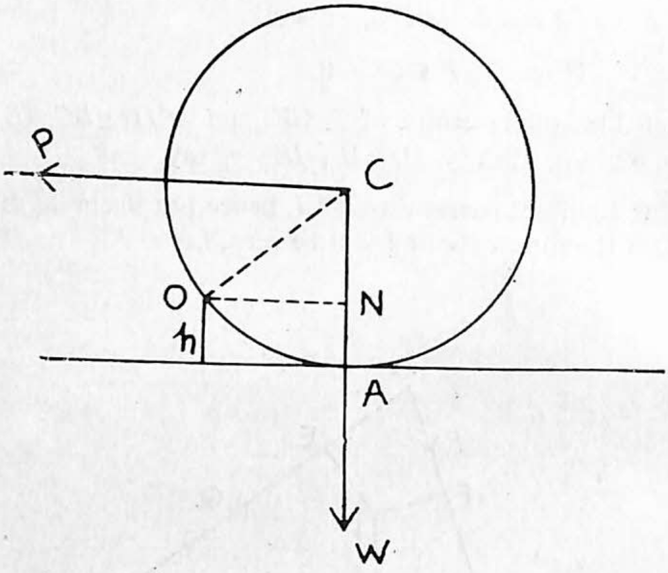


FIG. 63

To overcome the obstacle, the moment of P about O must be slightly greater than that of W about the same point. In the limiting position of equilibrium in which the wheel is about to turn around O, the pressure of the ground through A will be just zero. Hence, taking moment of P and W about O, we get

$$P.\,NC > W.ON$$

$$\Rightarrow P.\,(\mathrm{r}-\mathrm{h}) > W.\,\sqrt{OC^2-CN^2}$$

$$= W.\,\sqrt{\mathrm{r}^2-(\mathrm{r}-\mathrm{h})^2}$$

$$= W\sqrt{2\mathrm{hr}-\mathrm{h}^2}$$

$$\Rightarrow P > \frac{W\sqrt{2\mathrm{hr}-\mathrm{h}^2}}{\mathrm{r}-\mathrm{h}}.$$

Hence the result.

Ex. 8. Of the four coplanar forces in equilibrium, one is given completely, a second and a third, which are not parallel, have their lines of action given, while the fourth has its magnitude only given. Prove that the line of action of the fourth force must touch a fixed circle.

[C.U. 1934]

Let the four forces be *P*,*Q*,*R* and *S* of which *P* is completely given i.e. its magnitude, direction and sense are known, the lines of action of the non-parallel forces, *Q* and *R* intersecting at *O* are given and the magnitude only of *S* is given. To prove that the line of action of the force *S* touches a fixed circle. Let s and p be the lengths of the perpendiculars drawn from *O* on the lines of action of *S* and *P* respectively. Since *P* is completely given and the lines of action of *Q* and *R* are given, *P* and p are given constants but s is unknown. Now, since the system of forces is in equilibrium, the algebraic sum of the moments of all the forces about any point in their plane is zero.

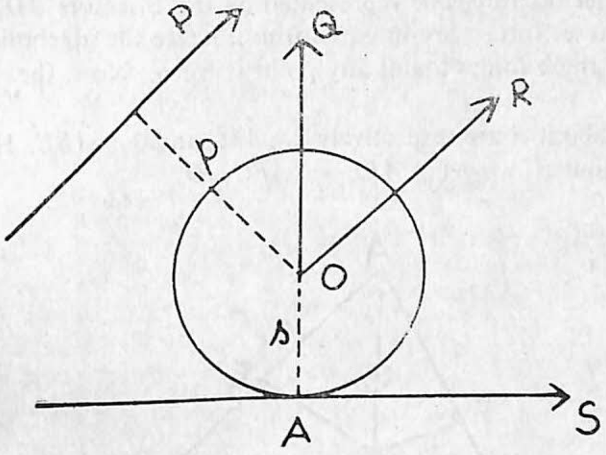


FiG. 64

Hence, taking moments about O, we get,

$$Pp - Ss = 0$$

$$\Rightarrow s = \frac{Pp}{S}.$$

which is a constant, since P, p and S are all given. Now, since the distance of the line of action of the forces S from a fixed point O is constant for all drections of the force of magnitude S, it follows that the line of action of S touches a fixed circle whose centre is O and radius is $\frac{Pp}{S}$. Hence the result.

Ex. 9. If three forces represented in magnitude and direction by the bisectors of the angles of a triangle, all acting from the vertices, be in equilibrium, prove that the triangle must be equilateral.

Let AD, BE and CF be the bisectors of the angles A, B and C of $\triangle ABC$ and let the forces be represented by the bisectors $\vec{AD}$, $\vec{BE}$ and $\vec{CF}$. Since all these forces are in equilibrium, hence the algebraic sum of the moments of these forces about any point is zero. Now, the moments of $\vec{CF}$ and $\vec{BE}$ about A are respectively $2\triangle AFC$ and $2\triangle ABE$. Hence, taking moments about A, we get $\triangle ABE - \triangle AFC = O$ ...(1)

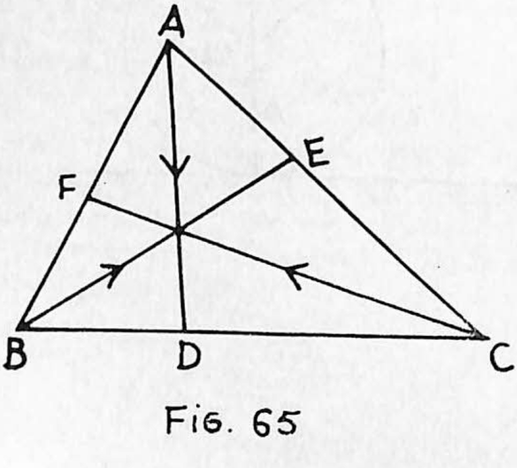


Fig. 65

Now, since BE is the bisector of the angle B, hence,

$$\frac{\triangle BCE}{\triangle BAE} = \frac{CE}{AE} = \frac{BC}{AB} = \frac{\mathrm{a}}{\mathrm{c}}$$

$$\Rightarrow \frac{\triangle BCE + \triangle BAE}{\triangle BAE} = \frac{\mathrm{a+c}}{\mathrm{c}} \Rightarrow \frac{S}{\triangle BAE} = \frac{\mathrm{a+c}}{\mathrm{c}},$$

where S is the area of $\triangle ABC$.

Hence, $\triangle\ ABE = \dfrac{\mathrm{c}}{\mathrm{a+c}} S.$

Similarly, $\triangle\ AFC = \dfrac{\mathrm{b}}{\mathrm{a+b}} S.$

Hence, from (1),

$$\frac{\mathrm{c}}{\mathrm{a+c}} - \frac{\mathrm{b}}{\mathrm{a+b}} = 0$$

$$\Rightarrow \mathrm{c(a+b)-b\ (a+c)=0}$$

$$\Rightarrow \mathrm{a\ (b-c)=0 \Rightarrow b=c} \quad [\because \mathrm{a} \neq 0].$$

Similarly, taking moments about B, we can show that a=c.

Hence, a=b=c i.e the given triangle is equilateral.

Ex. 10. OX and OY are two straight lines at right angles, and a force acting in their plane at O has moments G and G' about the two points whose co-ordinates are (x,y) and (x′, y′) respectively with respect to the lines OX and OY as axes of co-ordinates. If $\mathrm{xy'-x'y} \neq 0$, prove that the magnitude R of the force and the angle θ between its line of action and OX are given by

$$R^2 = \frac{(\mathrm{x}G' - \mathrm{x}'G)^2 + (\mathrm{y}G' - \mathrm{y}'G)^2}{(\mathrm{xy'-x'y})^2}$$

and $\tan\ \theta = \dfrac{\mathrm{y}\ G' - \mathrm{y}'\ G}{\mathrm{x}\ G' - \mathrm{x}'\ G}.$

Let P and P' be the points (x,y) and (x′, y′) respectively. The two components of R along OX and OY are $R \cos\theta$ and $R \sin\theta$ respectively. Since the moments of a force about any point is equal to the algebraic sum of the moments of their components about the same point, we have by taking moments about P and P′ successively,

$$R \cos\theta.\ \mathrm{y}' - R \sin\theta.\ \mathrm{x} - G = 0 \qquad \ldots(1)$$

$$R \cos\theta.\ \mathrm{y} - R \sin\theta.\ \mathrm{x}' - G' = 0. \qquad \ldots(2)$$

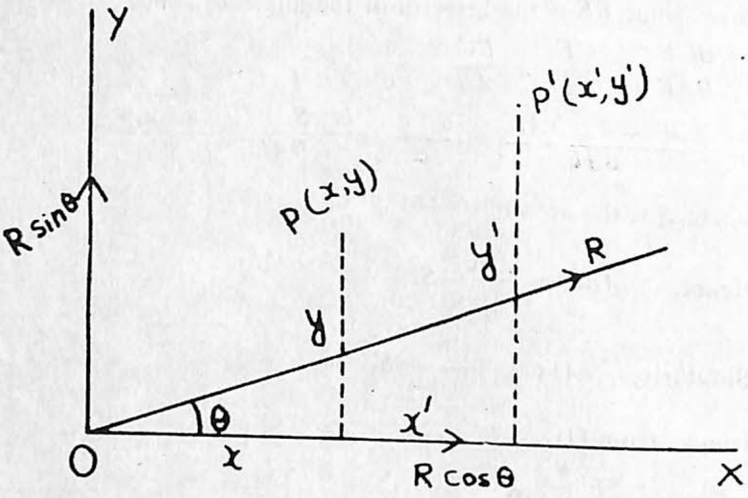


FIG. 66

Solving for $R\cos\theta$ and $R\sin\theta$, we get

$$\frac{R\cos\theta}{xG'-x'G}=\frac{R\sin\theta}{-Gy'+G'y}=\frac{1}{-yx'+xy'}$$

$$R\cos\theta=\frac{xG'-x'G}{xy'-x'y} \text{ and } R\sin\theta=\frac{yG'-y'G}{xy'-x'y}$$

Squaring and adding, we get

$$R^2=\frac{(xG'-x'G)^2+(yG'-y'G)^2}{(xy'-x'y)^2}.$$

Dividing the second equation by the first, we get

$$\tan\theta=\frac{yG'-y'G}{xG'-x'G}.$$

Hence the result.

Ex. **11**. *ABC* is an isosceles right-angled triangle whose equal sides *AB* and *AC* are 4cm in length. The moments of a force about the points *A*, *B* and *C* are respectively 8 *N*m, 8 *N*m and 16 *N*m in the same sense. Find the magnitude and the line of action of the force.

Let the force be *F*. Since the moments of *F* about *A*, *B* and *C* are in the same sense, *A*, *B* and *C* must be on the same side of the line of action of *F*. If p_1 and p_2 be the lengths of the perpendiculars from *A* and *B* respectively on the line of action of *F*, we have, by taking moments about *A* and *B*,

$Fp_1 = 8$ and $Fp_2 = 8$

from which we get $p_1 = p_2$. Hence, the line of action of F is parallel to AB. We produce CA to cut the line of action of F at N. Taking

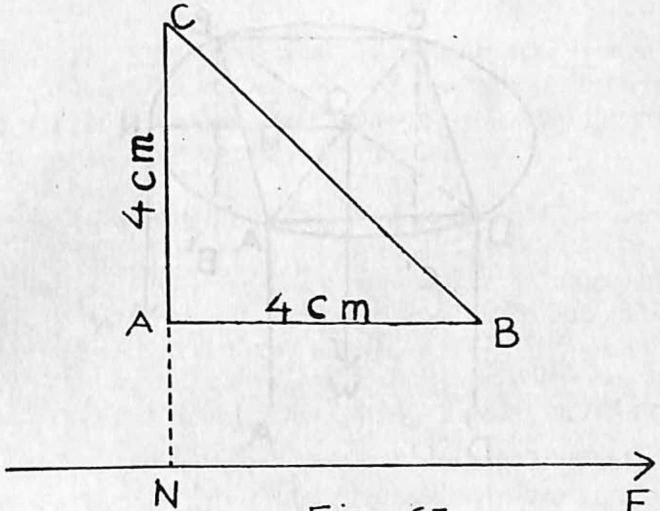


FIG. 67

moment about C, we get

$F.\ CN = 16$

and $F.\ AN = 8$

Hence, $\frac{CN}{AN} = 2 \Rightarrow CN = 2\ AN.$

Hence, $AN = AC = 4.$

$\therefore F = \frac{8}{4} = 2.$

Hence, the magnitude of the force is $2N$ acting parallel to AB at a distance of 4 cm from it on the opposite sides of C.

Ex. 12. A circular table of weight W has four legs spaced at equal distances round its edge. Show that the least weight sufficient to overturn the table is $(\sqrt{2}+1)W$.

Let $ABCD$ be a circular table and let A, B, C and D be the upper ends of the legs attached to the rim such that $ABCD$ is a square. Let A', B', C' and D' be the points of contact of the legs with the floor. O is the centre of the table, through which its weight, W, acts vertically downwards. If any weight is placed within the square, there is no chance of overturning, because, in this case the moment of this weight about any side of the square will be of the same sign as that of the weight of the table about the same side. The only case in which the table has any chance of overturning is when the weight is placed on the table outside the square, say, in the portion *ANB*. Now, the weight placed will have the greatest turning effect when placed farthest away from *AB* i.e. at *N* where *OMN* is perpendicular to *AB*. Clearly, *N* is the middle point of the arc *AB* and $\angle OAM = 45°$. Let W' be the weight placed at *N*. Now, when the table is just on the point of being overturned about *AB*, the legs at C' and D' will just lose contact with the ground and *W* and W' will have equal moments about the line $A'B'$ i.e. about *AB*. Hence, taking moment about *AB*, we get

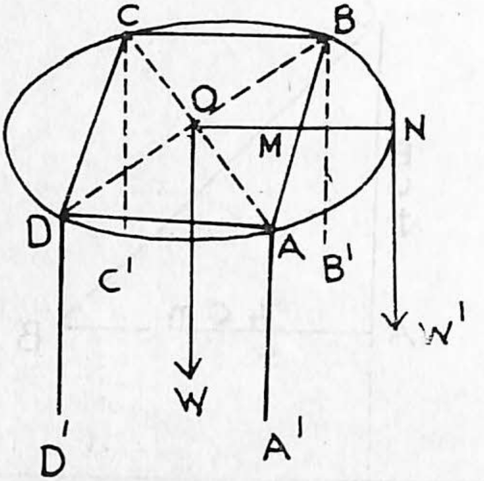


FIG. 68

$$W.\, OM = W'\, MN$$

$$\Rightarrow W \text{a} \sin 45° = W'. (\text{a} - \text{a} \sin 45°),$$

where a is the radius of the circular table.

$$\text{Hence, } W' = \frac{W.\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{W}{\sqrt{2} - 1} = W(\sqrt{2} + 1)$$

EXERCISE 5.1

1. A uniform rod 50 cm long and weighing 50 N rests horizontally on supports at its ends. If a weight of 15 N is attached at a point 30 cm from one end, find the pressures on the supports.

[Ans. 31 N and 34N]

2. Weights of 5N, 10N, 15N and 20N are suspended from a uniform rod of length 2m and weight 15 N at distances 40cm, 70cm, 1m and 1m 20 cm respectively from one end. Find the position of the point about which the rod will balance.

$\left[\text{Ans. } 96\frac{12}{13}\text{cm from the end from which distances are measured.}\right]$

3. Find the magnitude and line of action of the resultant of parallel forces of magnitudes 3, 6, 8 units in one direction and 12 in the opposite direction acting at points A, B, C and D respectively in a straight line, where $AB=1$m, $BC=3$m and $CD=5$m.

[Ans. 5 units at 14 m from A on DA produced.]

4. A heavy uniform beam of length 4m and weight 300 N is suspended in a horizontal position by two vertical strings attached at the ends, each of which can just sustain a tension of 200N. Within what distance from the centre can a weight of 75N be suspended without breaking either string?

$\left[\text{Ans. } \frac{2}{3}\text{ m}\right]$

5. A uniform rod of length 2m and weight 10N rests in a horizontal position with its ends on two supports each of which will bear a weight of 60 N and no more. Find at what part of the rod a weight of 85 N can be placed without breaking either support.

$\left[\text{Ans. } \frac{5}{17}\text{m on either side of the rod.}\right]$

6. Forces of magnitudes 1N, 2N, 3N, 4N, 5N and 6N act along the sides AB, BC, CD, DE, EF and FA of a regular hexagon. Find the moment of each force about A.

$\left[\text{Ans. } 0,\ a\sqrt{3}, 3a\sqrt{3}, 4a\sqrt{3}, 5a\frac{\sqrt{3}}{2} \text{ and } 0, \text{ where } a=\text{the side of the hexagon.}\right]$

7. Five parallel forces 1,6,3,4 and 8 units act on 1m apart on a straight horizontal rod. What force must be added to the first force in order that, if the rod be supported at the point where the third force acts, it may remain horizontal.

[Ans. 6 units]

8. The horizontal roadway of a bridge AB, 12m long, weighs 56000 N and rests on two supports at its ends. What is the pressure on each support when a lorry of weight 33600N starting from A is two-thirds of the way across the bridge?

[Ans. 39200N at A and 50400N at B]

9. A uniform plank ABC of length 12m and weight 800N rests on two supports A and B, one at the end A and the other at B, $4\frac{1}{2}$ m from the end C. A boy walks along the plank from A to C and just as he reaches C, the plank commences to tilt. Find the weight of the boy.

[Ans. $266\frac{2}{3}\,N$]

10. A light rod $ABCD$, 2m long, has a vertical force 75 N acting at its middle point. It rests on two props B and C, 1m apart, and the pressure on C is 4 times that on B. Find the greatest vertical force that can act at D without overturning the rod.

[Ans. $18\frac{3}{4}\,N$]

11. A stiff pole 4m long sticks horizontally out from a vertical wall. It would break if a weight of 140 N were hung at the end. How far out along the pole may a boy weighing 560N venture with safety?

[Ans. The boy can go a distance just less than 1m from the wall.]

12. A uniform beam 4m long and weighing 300 N rests on two props at equal distances from the ends. Find the maximum value of this distance so that a man weighing 700N may stand anywhere on the beam without upsetting it.

[Ans. $\frac{3}{5}$m]

13. A uniform rod of length 3m and weight 20N is suspended horizontally by two strings, one attached at a point distant 1m from one end and the other attached at a point distant $\frac{1}{3}$m from

the other end. The maximum tension which either string can bear is $60N$. At what part of the rod a weight of $80N$ can be attached without either string breaking?

[Ans. If xm be the required distance from the first end, then $1\frac{17}{24} \leqslant x \leqslant 2\frac{1}{8}$]

14. A rod AB, 12 cm long, has a weight of $1N$ hung from A and when a weight of $15\ N$ is hung from B, it balances at a point 3 cm from B while if $8N$ is hung there, it balances at a point 4 cm from B. Find the weight of the rod and the point where it acts.

[Ans. $12N$, middle point]

15. A non-unifrom rod of length $(a+b)$ and weight W rests on two parallel knife edges at a distance c apart in the same horizontal plane, so that equal portions of the rod project beyond each knife edge. Prove that the pressures on the knife edges are respectively $\frac{a-b+c}{2c}W$ and $\frac{b-a+c}{2c}W$, if the weight of the rod acts at a point at a distance a from A.

16. A light horizontal rod, 12 cm long, is supported by two vertical props, each 3 cm from an end of the rod and is loaded with $16N$ at each end. What weights hung from the ends will produce in one prop a pressure double and in the other prop a pressure half of that produced by the $16N$ weights?

[Ans. $26N$ and $14N$]

17. A horizontal beam $ABCD$ rests on two supports at B and C where $AB=BC=CD$. It is found that the beam will just tilt when a weight of p N is hung from A or when a weight of q N is hung from D. Find the weight of the beam and prove that the point on the beam, where the weight acts, divides AD in the ratio $2p+q : p+2q$.

[Ans. $(p+q)\ N$]

18. A uniform bar, $3\frac{1}{2}$ m long and weighing $6N$, rests on a horizontal table with one end projecting $1\frac{1}{2}$m over the edge. Find the greatest weight that can be hung on the end without making the bar topple over.

19. A non-unifrom rod, 16 cm long, rests on two pegs 9 cm apart, with its centre midway between them. The greatest weights that can be suspended in succession from the two ends without

disturbing the equilibrium are $4N$ and $5N$ respectively. Find the weight of the rod and the position of the point at which its weight acts.

[Ans. $3\frac{1}{2}N$, $\frac{1}{2}$ cm from the mid-point]

20. Prove that the line of action of the forces $11N$ acting along DA, $7N$ along CB, $19N$ along CD and $5N$ along BA of a square $ABCD$ bisects AD and trisects CD.

21. Three forces, each of magnitude P, act along the sides of a triangle ABC in order. Prove that the resultant is at a distance of $\frac{2P\triangle}{aR}$ from A cutting BC externally in the ratio c : b, where R is the resultant of the three forces, $\triangle$ is the area of the triangle ABC and a,b,c are the lengths of BC, CA and AB respectively.

22. ABC is an equilateral triangle. Forces $4N$, $2N$ and $1N$ act along the sides AB, AC and BC respectively in the directions indicated by the letters. Prove that their resultant acts in a direction at right angles to BC and find the point in which the line of action meets BC.

[Ans. The line of action divides BC in the ratio 1 : 2.]

23. A heavy non-uniform beam AB of length l rests across a fixed peg P and carries equal weights W_1 hung from each end. In the equilibrium position with the beam horizontal, the distance AB is x_1. For another pair of equal weights W_2, the corresponding distance AP is x_2. Prove that the weight of the beam, W, is given by $(x_1-x_2)W=(W_1-W_2)x_2\,l-2W_1x_1+2W_2x_2$.

24. A uniform plank of length 2a and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respeetively. Show that

$$\frac{W_1}{W+W_1}+\frac{W_2}{W+W_2}=\frac{b}{a}.$$

25. One end of a stout rope of length 20 m is fixed to a vertical telegraph post standing on the ground, and a man pulls at the other end with a given force. Find the highest point of the post at which the rope is to be fixed in order that the man will have the least chance of over-turning the post.

[Ans. The rope is to be fixed at a height of $10\sqrt{2}$ m from the ground.]

26. A smooth bamboo pole just stands vertically on the ground, and a horizontal rope which is once wrapped at its top has the two portions at right angles to one another. The pole is kept in position by pulling it with a rope attached at one-third the height of the pole. If this latter rope be inclined at an angle 45° with the horizon, prove that the tension in it must be six times that of the rope at the top.

27. The sums of the moments of a system of forces acting at a point about two given points are equal in magnitude. Show that their resultant is parallel to a fixed line or passes through a fixed point.

28. The moments of a force about the points (0,0), (10,0) and (0,5)n are 184 Nm,$-46N$m and 249 Nm. Find where the force meets the x-axis and find its components parallel to the co-ordinate axes.

    [Ans. At a distance 8m from the origin. 13N along x-axis an 23N along y-axis]

29. Forces are represented in magnitude, direction and line of actiod by the perpendiculars drawn from the angular points of a triangle to the opposite sides. If their sum of moments about each of the angular points is zero, show that the triangle is equilateral.

30. The wire passing round a telegraph pole is horizontal and the two portions attached to the pole are inclined at an angle of 60° to one another. The pole is supported by a wire attached to the middle point of the pole and inclined at 60° to the horizon. Show that the tension of this wire is $4\sqrt{3}$ times that of the telegraph wire.

31. AB is a diameter of a circle and BP and BQ are chords at right angles to one another. Show that the moments of the forces represented by $\overrightarrow{BP}$ and $\overrightarrow{BQ}$ about A are equal.

32. M_1 and M_2 are the moments of a force F about two given points A_1 and A_2 respectively, where M_1 and M_2 are of the same sign. Prove that the line of action of the force is along a common tangent to two circles drawn with centres A_1 and A_2 and radii $\frac{M_1}{F}$ and $\frac{M_2}{F}$ respectively.

33. If the moments of two given intersecting forces about a point in their plane be equal and of the same sense, prove that the point must be on a certain straight line.

34. Forces $1N$, $2N$, $4N$ and $5N$ act, all in the same sense, along the sides of a square taken in order. Prove that their resultant is parallel to a diagonal and find where it cuts the side along which the first force acts.

[Ans. It divides the side externally in the ratio 2:3.]

35. A letter weigher consists of a uniform plate in the form of a right-angled isosceles triangle ABC of mass $\frac{3}{32}$ kg, which is suspended by its right angle C from a fixed point to which a plumb-line is also attached. The letters are suspended from the angle A, and their masses read off by observing where the plumb-line intersects a scale engraved along AB, the divisions of which are marked $\frac{1}{32}$ kg, $\frac{2}{32}$ kg, $\frac{3}{32}$ kg, etc. Show that the distance from A of the divisions of the scale form a harmonic progression.

36. ABC is a right-angled triangle, the sides BC, CA and AB being 13m, 12m and 5m of length respectively. The moments of a force F about A,B and C are respectively

(i) 0 Nm, 25 Nm and 144 Nm respectively

(ii) 0 Nm, -25 Nm and 144 Nm respectively.

Find in each case the magnitude, direction and the line of action of F.

[Ans. (i) $F=13N$ acting along the tangent at A to the circum-circle of $\triangle ABC$.

(ii) $F=13N$ acting along the perpendicular from A on BC.]

37. Three forces P, Q and R act along BC, CA and AB respectively of a triangle ABC. If the line of action of their resultant passes through

(i) the in-centre and the circum-centre of $\triangle ABC$, show that $P:Q:R=(\cos B-\cos C):(\cos C-\cos A):(\cos A-\cos B)$ $=(b-c)(b+c-a):(c-a)(c+a-b):(a-b)(a+b-c)$

(ii) the in-centre and the centroid of $\triangle ABC$, show that $P:Q:R=a(b-c):b(c-a):c(a-b)$

(iii) the in-centre and the ortho-centre of $\triangle ABC$, show that $P:Q:R=\cos A\,(\cos B-\cos C):\cos B\,(\cos C-\cos A)$ $:\cos C(\cos A-\cos B)$

(iv) the circum-centre and the ortho-centre of $\triangle ABC$, show that
$P : Q : R = \sin 2A \sin (B-C) : \sin 2B \sin (C-A)$
$: \sin 2C \sin (A-B) = (b^2-c^2) \cos A :$
$(c^2-a^2) \cos B : (a^2-b^2) \cos C$

(v) the circum-centre and the centroid of $\triangle ABC$, show that
$P : Q : R = \sin 2A \sin (B-C) : \sin 2B \sin (C-A)$
$: \sin 2C \sin (A-B)$

(vi) the ortho-centre and the centroid of $\triangle ABC$, show that
$P : Q : R = \sin 2A \sin (B-C) : \sin 2B \sin (C-A) :$
$\sin 2C \sin (A-B)$,
where a, b, c are the sides of the triangle ABC.

38. Three forces P, Q and R act along the sides BC, CA and AB respectively of a triangle ABC. If their resultant passes through the centroid of the triangle, prove that

$$\frac{P}{BC} + \frac{Q}{CA} + \frac{R}{AB} = 0.$$

39. Like parallel forces P, Q and R act at the vertices of a triangle ABC perpendicular to its plane. If the resultant passes through

(i) the in-centre of $\triangle ABC$, show that
$P:Q:R = \sin A : \sin B : \sin C$

(ii) the circum-centre of $\triangle ABC$, show that
$P:Q:R = \sin 2A : \sin 2B : \sin 2C$
or, $P(c \cos B - b \cos C) + Q(a \cos C - c \cos A)$
$+ R(b \cos A - a \cos B) = 0$

(iii) the ortho-centre of $\triangle ABC$, show that
$P:Q:R = \tan A : \tan B : \tan C$
or, $P(b^2+c^2-a^2) = Q(c^2+a^2-b^2) = R(a^2+b^2-c^2)$.

40. Three forces P, Q and R acting at the vertices A, B and C respectively of a triangle ABC, each perpendicular to the opposite side, keep it in equilibrium; prove that
$P:Q:R = a:b:c$.

41. If three forces P, Q and R acting along the bisectors of the angles of $\triangle ABC$ at the angular points A, B and C respectively, keep the triangle in equilibrium; show that
$P:Q:R = \cos \frac{1}{2} A : \cos \frac{1}{2} B : \cos \frac{1}{2} C$.

42. Three forces P, Q and R act along the sides BC, CA and AB of a triangle ABC. Show that the line of action of their resultant

cuts BC at a point distant x from C, and BA at a point distant y from A, given by

$$x=\frac{\mathrm{ab}\,R}{Q\mathrm{c}-R\mathrm{b}},\ y=\frac{\mathrm{bc}\ P}{Q\mathrm{a}-P\mathrm{b}}.$$

43. Three forces acting along the medians of a triangle, all from the vertices, are in equilibrium. Show that the forces are proportional to the lengths of the medians.

44. Forces $l.BC$, $m.CA$ and $n.\ AB$ act along the sides of a triangle ABC, taken in order. Show that their resultant passes through the centroid of the triangle if $l+m+n=0$.

45. Forces P, Q and R act along the sides BC, CA and AB respectively of a traingle ABC, and the forces P', Q' and R' act along OA, OB and OC respectively, where O is the circum-centre of $\triangle ABC$. Prove that, if the six forces are in equilibrium,

$$P\cos A+Q\cos B+R\cos C=0$$

and $$\frac{PP'}{\mathrm{a}}+\frac{QQ'}{\mathrm{b}}+\frac{RR'}{\mathrm{c}}=0.$$

46. Three forces act along the sides of a triangle, taken in order. If the sum of two of the forces be equal in magnitude but opposite in sense to the third force, prove that their resultant passes through the in-centre of the triangle.

47. Three forces P, Q and R act along the sides BC,CA and AB respectively of a triangle ABC, taken in order. If their resultant touches the

   (i) circum-circle, show that

   $$\mathrm{a}P+\mathrm{b}Q+\mathrm{c}R=0$$

   (ii) in-circle, show that

$$\frac{\cos^2\frac{A}{2}}{P}+\frac{\cos^2\frac{B}{2}}{Q}+\frac{\cos^2\frac{C}{2}}{R}=0.$$

48. If the four forces, each acting along a cyclic quadrilateral, be in equilibrium, show that each force is proportional to the opposite side.

49. A square table stands on four legs placed at the mid-points of its sides. If the total weight of the table and legs be W, find the

greatest weight which can be put at one of the corners of the table without upsetting it.

[Ans. W].

50. A round table of weight W stands on three legs, of which the upper ends are attached to its rim so as to form an equilateral triangle. Show that a body whose weight does not exceed W may be placed anywhere on the table without the risk of toppling it over.

Chapter VI

Couple and its moment

6.1 Definitions

We have seen that two equal and unlike parallel forces cannot be combined into a single resultant. Such forces acting on a rigid body cannot, therefore, produce any translatory motion of the body. They can generate only the rotation of the body. This type of rotatory motion of a body due to two equal and unlike parallel forces is observed when we wind a clock or spin a spindle. The key or the spindle rotates due to two such forces which are generated by our fingers.

Two equal and unlike parallel forces whose lines of actions are different are said to constitute a couple. The effect of a couple on a body is to produce pure rotation.

The arm of a couple is the perpendicular distance between the lines of action of the two forces which form the couple.

The moment of a couple is the product of any one of the forces which form the couple and the perpendicular distance between their lines of action.

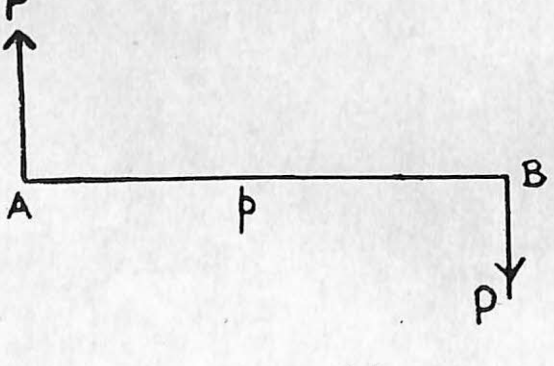


FiG. 69

Thus in fig. 69, AB is the arm of the couple formed by two equal and unlike forces P,P and p is the perpendicular distance between the lines of action of the forces.

The moment of a couple is regarded as *positive* or *negative* according as the couple tends to rotate the body in the anticlockwise or clockwise direction. Thus in fig. 69, the moment of the couple is $-Pp$, since the couple tends to rotate the body in the clockwise sense.

Remark : A couple with forces P,P and arm length p is very often denoted by (P, p).

A couple is also called *Torque* by some writers. Some, however, use the word 'torque' to denote the moment of the couple.

6.2 Theorem on the moment of a couple

The algebraic sum of the moments of the two foreces forming a couple about any point in their plane is a non-zero constant and equal to the moment of the couple.

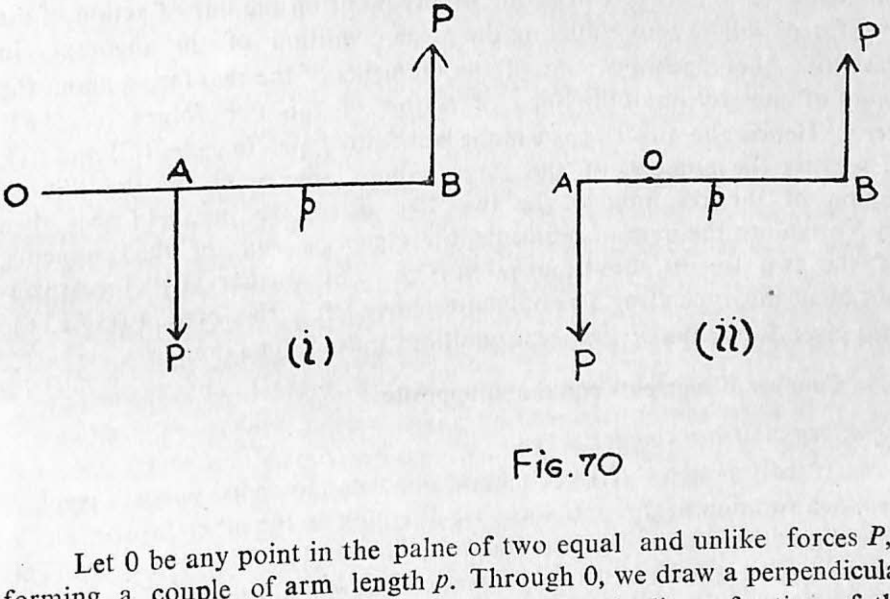


Fig.70

Let 0 be any point in the palne of two equal and unlike forces P,P forming a couple of arm length p. Through 0, we draw a perpendicular line OAB [fig. 70(i)] or AOB [fig. 70(ii)] meeting the lines of action of the forces at A and B. Then $AB=p$. In fig. 70 (i) where the point O lies outside the forces, the algebraic sum of the moments of the forces about $O=P.OB-P.OA=P(OB-OA)=P.AB=Pp$ which is independent of the position of the point O and hence it is a constant. This is also equal to the moment of the couple.

In fig. 70(ii), where the point O lies inside the equal forces, the algebraic sum of the moments of the forces about $O = P.OA + P.OB = P.(OA+OB) = P.AB = Pp$ which is also a constant and is equal to the moment of the couple.

Remark : The moment of a couple can never be zero, since both P and $p \neq 0$.

6.3 Converse theorem

The converse of the above theorem is also true i.e. *if the algebraic sum of the moments of two coplanar forces acting on a body about any point in their plane is a non-zero constant, then the forces form a couple.*

The two coplanar forces may be of any one of different types viz. (i) they may act along the same straight line, (ii) they may intersect at a point, forming a non-zero angle, (iii) they may be like parallel forces (iv) they may be unequal and unlike parallel forces or (v) they may be equal and unlike parallel forces in which case they form a couple. Now, if the forces are of the type (i), then the algebraic sum of the moments of the two forces about any point on the line of action of the two forces will be zero, violating the given condition of the theorem. In case (ii), the algebraic sum of the moments of the two forces about the point of intersection of the lines of action of the two forces is clearly zero. Hence, the two forces cannot be of this type. In cases (iii) and (iv), if we take the moments of the forces about any point on the line of action of the resultant of the two like or unlike parallel forces, then by Varignon's theorem of moments, the algebraic sum of the moments, of the two forces about that point is zero. So the two given forces cannot be of this type also. The only alternative left is the force in (v) i.e. the two given forces under the give conditions must form a couple.

6.4. Couples of moments equal and opposite

Equilibrium of two couples :

If two couples have equal and opposite moments, one will produce as much rotation in the anticlockwise direction as the other in the clockwise direction. Hence, the two couples will cancel each other's effect. This theorem of equilibrium of two couples can be stated as follows :

If two couples whose moments are equal in magnitude and opposite in sign act in the same plane upon a rigid body, they balance one another.

To prove this theorem. we consider two couples (P, p) and (Q, q) such that $P\text{p} = Q\text{q}$ .. (1)
in magnitude

There are two cases to be considered.

Case I. *When the forces forming the couples are not all parallel*

Let one of the forces of the couple (P,p) intersect one of the forces of the couple (Q,q) in A and the other two forces of the two couples intersect at B [fig. 71].

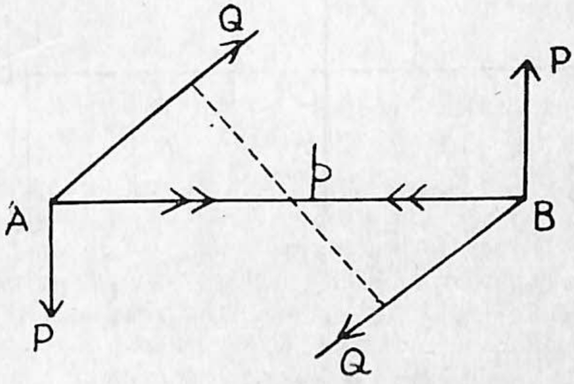


FIG. 71

Now, the algebraic sum of the moments of the forces P and Q acting at B about the point A

$= \text{Pp} - \text{Qq} = 0$ [from (1)].

Similarly, the algebraic sum of the moments of the forces P and Q acting at A about the point B is zero. Hence, it follows that the resultant of the forces P and Q acting at A passes through B and that of P and Q acting at B passes through A.

Now, the angle between P and Q at A is equal to that between P and Q at B and so the resultant of the forces P and Q at A must be equal in magnitude to that of P and Q at B. Since these two equal forces are acting along the same line in opposite directions AB and BA, they will balance each other. Thus the resultant of all the forces forming the two couples is zero and hence the two couples balance each other.

case II. *When the forces forming the couples are all parallel*

We draw a straight line perpendicular to the lines of action of the forces forming the two couples (P,p) and (Q,q) to meet them at A, A_1, B and B_1. Here $\text{p}=AB$ and $\text{q}=A_1B_1$. Let the resultant of the like parallel forces Q at A_1 and P at B pass through C such that

$$Q.\, A_1\, C = P.\, BC \qquad \dots(2)$$

The magnitude of this resultant is $P+Q$ and the line of action of this resultant is perpendicular to $A\,B_1$.

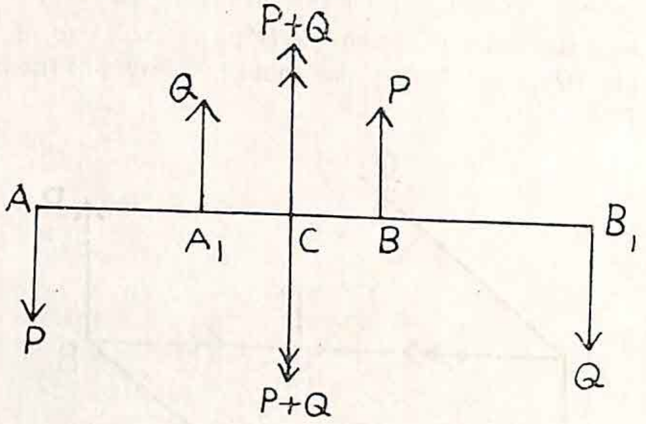


FIG. 72

Now, since the moments of the two couples are equal in magnitude, we have Pp=Qq

$\Rightarrow Q.\ A_1\ B_1 = P.\ A\ B$.........(3)

Subtracting (2) from (3), we get

$Q.\ (A_1B_1 - A_1C) = P.\ (AB - BC)$

$\Rightarrow Q.\ B_1C = P.\ AC.$

This shows that the resultant of Q at B_1 and P at A passes through C. Its magnitude is also $P+Q$ and its direction is perpendicular to AB_1.

Thus the resultant of P at B and Q at A_1 and that of P at A and Q at B_1 are equal in magnitude and act along the same straight line in the opposite directions. Hence, they balance each other. Thus the two given couples also balance each other.

6.5 Equivalent couples

Theorem : Two couples in the same plane whose moments are equal and of the same sign are equivalent to one another.

If we reverse the constituent forces of any one of the two couples, then this couple and the remaining couple will balance each other, since their moments are now equal in magnitude but opposite in sign. It follows that the two coplanar couples whose moments are equal and of the same sign must be equivalant to one another.

Remark 1 : From the above theorem it follows that a couple can be replaced by any other couple acting in the same plane provided the two couples have the same moments and are of the same sign. It is immaterial what the direction of the constituent forces of the second couple may be or their magnitude or the arm. Thus a couple (P,p) can be replaced by a couple $\left(P_1, \frac{\text{Pp)}}{P_1} \right)$ in the same plane with its constituent forces each equal to P_1. The arm length of this couple should be such that the two moments are equal and of the same sign. Similarly, a couple (P,p) may be replaced by a couple $\left(\frac{P\text{p}}{\text{x}}, \text{x}\right)$ with a given arm length x anywhere in the same plane.

Remark 2 : *A* couple has no particular position of application like a force, but it may be transferred anywhere in its plane without changing its effect on the body it acts upon. We have thus the *principle of transmissibility of couples* just like the principle of transm issibility of forces.

6.6 Couples in parallel planes

Theorem : *The effect of a couple remains unchanged if it is transferred to a parallel plane without changing its moment in magnitude and sign.*

Let the couple (P, p) lie on the plane XY and let its arm length be AB so that p$=AB$ and the constituent forces are perpendicular to AB. Let $X' Y'$ be a plane parallel to the plane $X Y$. We take on $X' Y'$ any ine A_1B_1 such that $A_1B_1=AB$ and$A_1B_1 \parallel AB$. We join AB_1 and A_1B and let [them meet at O [fig. 73]. Clearly, O is the middle point of both AB_1 and A_1B. At each of the points A_1 and B_1, we introduce two equal and opposite forces, each being equal to P and parallel to its line of action. We can combine the equal and like parallel forces P, P acting at A and B_1 to a single resultant of magnitude $2P$, acting at O along OC parallel to each of the components. Similarly, the equal and like parallel forces P,P acting at A_1 and B can be replaced by their resultant $2P$, acting at O along OD parallel to the components. These two resultant forces of the same magnitude $2P$ act at O along the same line but in the opposite directions. Hence, they balance each other. We are now left with two equal and unlike parallel forcea P,p acting at A_1 and B_1, in the same sense and direction as those of the constituent forces of the couple (P,p) in the XY plane.

Thus the given couple (P,p) in the XY plane is equivalent to the couple (P,p) of the same moment in the parallel plane $X'Y'$ with arm A_1B_1. Since the couple in the $X'Y'$ plane can be replaced in its plane by any other couple of the same moment and sign, it follows that a couple

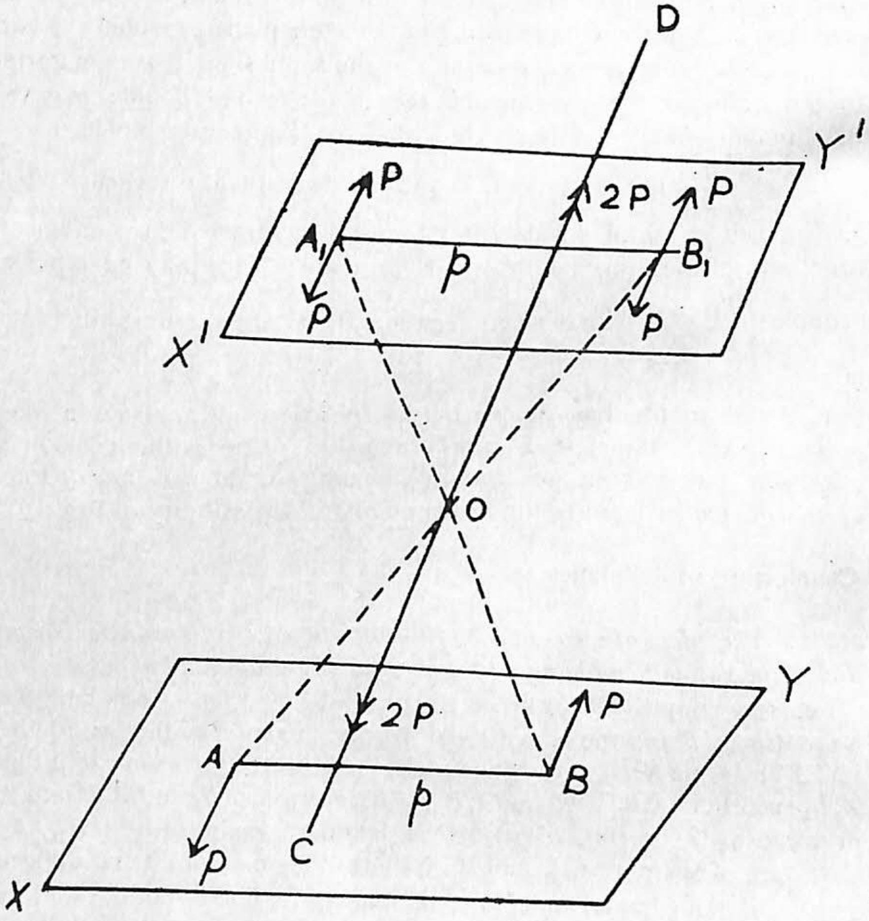


FIG. 73

in any plane can be replaced by any other couple in a parallel plane provided its moment remains unchanged in magnitude and sign.

Remark : *A* couple is completely specified if we know the direction of the set of parallel planes, the magnitude of its moment and the sense in which it acts. We can, therefore, represent a couple by a line segment drawn perpendicular to the set of parallel planes to indicate the direction, the length of the line segment will indicate the magnitude of its moment and the definite sense of the line segment will indicate the sense of the moment.

6.7 Resultant of a number of coplanar couples

We shall now prove the following theorem :

Theorem : Any number of coplanar couples acting on a body is equivalent to a single couple whose moment is equal to the algebraic sum of the moments of the couples.

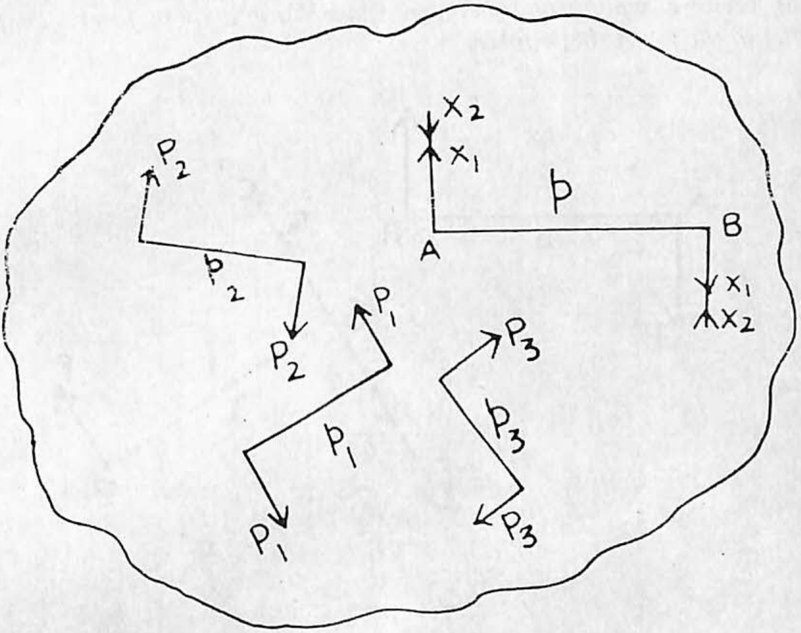


FiG. 74

Let the coplanar couples be $(P_1,\ p_1)$ (P_2,p_2) $(P_3,\ p_3)$,...... $(P_n,\ p_n)$ acting on a rigid body [fig. 70]. We replace all the couples by equivalent couples $(X_1,\ p)$, $(X_2,\ p)$, $(X_3,\ p)$......$(X_n,\ p)$ respectively all having a common arm say, $AB=p$, where

$$X_1=\frac{P_1p_1}{p},\quad X_2=\frac{P_2\,p_2}{p},\ X_3=\frac{P_3\,p_3}{p}=......X_n=\frac{P_n\ p_n}{p}.$$

The forces X_1, X_2,......X_n of these couples act at the extremities of the arm AB and perpendicular to AB. The resultant of the forces acting at A is their algebraic sum i.e. $X_1+X_2......+X_n$ in one direction and that of the forces acting at B is $X_1+X_2......+X_n$ in the opposite direction. These two resultants being equal, unlike and parallel form a couple whose moment is equal to

$$(X_1+X_2+\ldots\ldots+X_n)\,\text{p}$$
$$=P_1\text{p}_1+P_2\text{p}_2+\ldots\ldots+P_n\text{p}_n$$
= algebraic sum of the moments of the separate couples.

6.8 Resultant of a force and a couple

Theorem : A force and a couple acting in the same plane upon a rigid body cannot produce equilibrium but are equivalent to a single force, equal and parallel to the given single force.

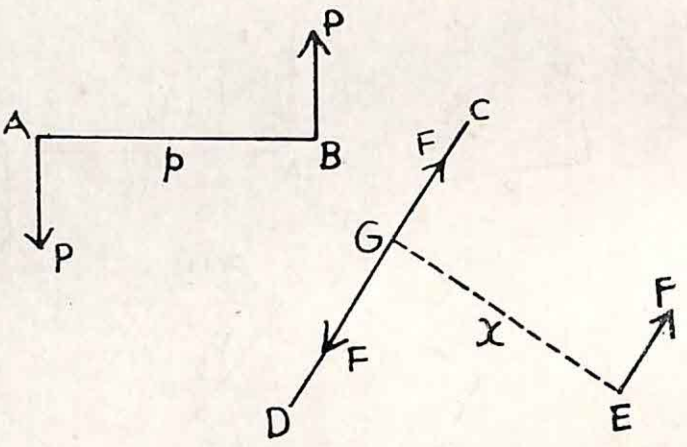


FIG. 75

Let (P,p) be the couple with arm length AB=p and let F be the given force acting at G. We shall show that the couple (P,p) and the single force F at G are equivalent to a force. We can replace the given couple by another couple whose constituent forces are F, F and the length of the arm is x where

$$\text{x} = \frac{P\text{p}}{F}.$$

We place this couple such that one of its component forces F acts at G along the line of action of the given force F but in the opposite sense along GD. Now, the two equal and opposite forces F, F acting at G along the same line will balance each other having a single force F at E, which is in the direction of the original force and at a distance of $\frac{P\text{p}}{F}$ from G. Thus a force and a couple in the same plane are equivalent to a single force and hence they cannot produce equilibrium.

6.9 Forces represented by the side of a triangle

We now prove a very important theorem which states as follows :
Theorem : If three forces acting on a rigid body be represented in magnitude, direction, and line of action by the sides of a triangle, taken in order, then they are equivalent to a couple whose moment is equal to twice the area of the triangle.

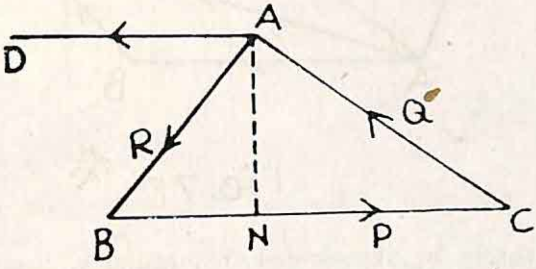


FIG 76

Let the three forces $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ of magnitudes P, Q and R respeetively act along the sides BC, CA and .AB of the traingle ABC [fig. 76].

Now, by triangle law of vectors, we have

$$\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$$

i.e. the resultant of the forces Q and R acting along AC and AB respectively is of magnitude P and will act at A in the direction AD parallel to CB. The force of magnitude P acting at A and another force of magnitude P acting along BC will form a couple of moment

$P \times AN = BC \times AN =$ twice the area of $\triangle$ ABC, where $AN \perp BC$. Thus the three forces represented by the sides of the triangle ABC form a couple of moment equal to twice the area of the triangle.

6.10 Forces represented by the sides of a polygon

As a generalisation of the above theorem, we now prove the following theorem :
Theorem : If a system of coplanar forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a polygon, taken in order, then they are equivalent to a couple whose moment is represented by twice the area of the polygon.

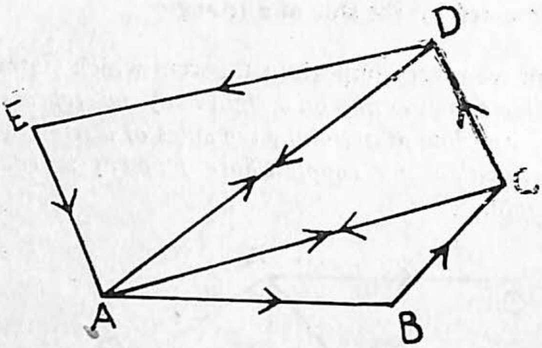


FiG. 77

Let the forces be represented in magnitude, direction and line of action by the sides *AB*, *BC*, *CD*, *DE* and *EA* of the polygon *ABCDE*. We join *AC* and *AD* [fig. 77]. We now introduce two pairs of equal and opposite forccs $\overrightarrow{AC}$, $\overrightarrow{CA}$ and $\overrightarrow{AD}$, $\overrightarrow{DA}$ acting along these lines. Since these forces balance each other, they will have no effect on the system. Now, forces $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ will form a couple of moment equal to twice the area of $\triangle$ *ABC*. Similarly, forces $\overrightarrow{AC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ will form a couple of moment equal to twice the area of $\triangle$ *ADC* and forces $\overrightarrow{AD}$, $\overrightarrow{DE}$ and $\overrightarrow{EA}$ will form a couple of moment equal to twice the area of $\triangle$ *ADE*. Now, these three couples are equivalent to a single couple whose moment is equal to twice the area of $(\triangle ABC + \triangle ACD + \triangle ADE)$ = twice the area of the polygon *ABCDE*. Hence the result.

6.11 Replacement of a force by the force and a couple

Theorem : *A force P acting at any point A of a body is equivalent to an equal and parallel force acting at any other arbitrary point B of the body together with a couple.*

Let the force *P* at *A* acting along *AE* be the given force and *B* is any arbitrary point. Let p be the length of perpendicular distance from *B* to the line of action of *P* at *A*. We apply equal and opposite forces at *B*, each equal and parallel to the force *P* at *A*. Let these two equal forces act along *BC* and *BD* along the same line. Since these two forces balance

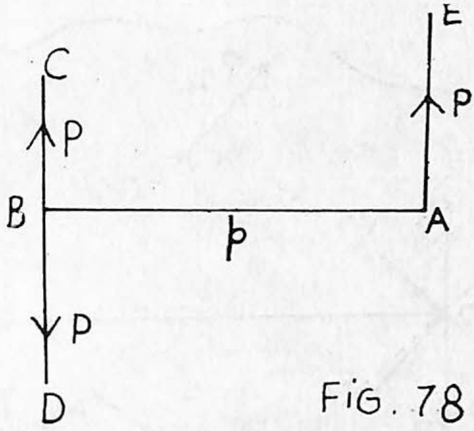


FiG. 78

each other, they will have no effect on the given system. The given force P at A can now be regarded as P along AE and P along BD forming a couple of moment $P\mathrm{p}$ and a force P at B along BC, parallel to the original force and in the same sense.

6.12 Moment of a system of forces forming a couple

Theorem : If a system of coplanar forces reduces to a couple, then the algebraic sum of the moments of the forces about any point in their plane is constant and equal to the moment of the couple.

Let $P_1, P_2 \ldots\ldots P_n$ be a system of coplanar forces acting at $A_1, A_2, \ldots A_n$ respectively of a rigid body and let O be any arbitrary point in their plane [fig. 79]. Let the perpendicular distances of the lines of action of the forces $P_1, P_2 \ldots\ldots P_n$ from O be $\mathrm{p}_1, \mathrm{p}_2, \ldots\ldots \mathrm{p}_n$ respectively. Then as in Section 6.11, we can replace the force P_1 by an equal and parallel force P_1 acting at O together with a couple of moment $P_1\mathrm{p}_1$. Dealing with each of the other forces in the same manner, we find that the given system of forces is equivalent to a system of concurrent forces $P_1, P_2, P_3, \ldots\ldots P_n$ at O, together with a number of couples. These couples can be combined into a single couple whose moment is

$P_1\mathrm{p}_1 + P_2\mathrm{p}_2 + \ldots\ldots + P_n\mathrm{p}_n$
= algebraic sum of the moments of the couples
$(P_1, \mathrm{p}_1), (P_2, \mathrm{p}_2), \ldots\ldots (P_n, \mathrm{p}_n)$ about O.

= algebraic sum of the moments of the given forces about O.

The set of concurrent forces at O must be in equilibrium, for, otherwise they would combine into a single resultant which, along with the resultant couple, would give us a single force as our resultant and not a conple.

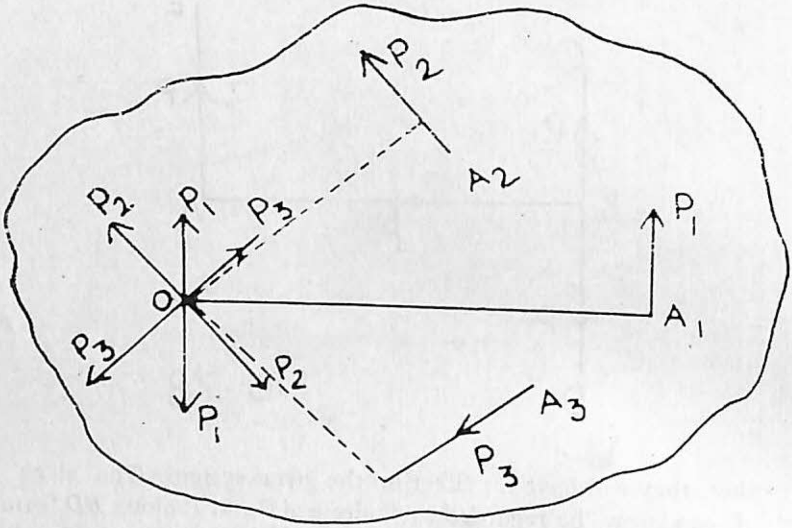


FIG. 79

Thus if a system of coplanar forces reduces to a couple, then the algebraic sum of the moments of the forces about any arbitrary point in their plane is constant and is equal to the moment of the resultant couple.

Remark 1: The converse of the above theorem can be stated as follows : *If three forces acting along the sides of a triangle be equivalent to a couple, they must be proportional to the sides along which they act in the same order*. This has been proved in Ex. 2 of Section 6.13.

Remark 2: The above theorem can be extended to any number of forces as stated below.

If any number of coplanar forces acting on a rigid body be represented in magnitude and direction and line of action by the sides of a polygon, taken in order, they are equivalent to a couple whose moment is represented by twice the area of the polygon.

6.13 Solved Examples

Ex. 1. Forces P, $2P$, $-P$ and $2P$ act along the sides AB, BC, CD and DA of the square $ABCD$, and a forces $P\sqrt{2}$ acts along eaah of BD and CA. Show that the forces reduce to a couple of moment $2aP$, where a is the side of the square.

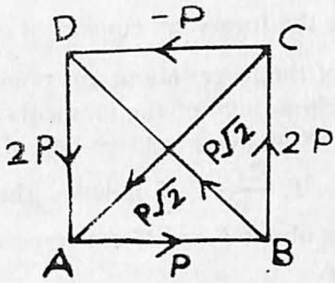


FiG. 80

We resolve the force $P\sqrt{2}$ acting along CA into two components, one $P\sqrt{2}\cos 45°$ i.e. P along CD and another $P\sqrt{2}\sin 45°$ i.e. P along CB. Similarly, we resolve the force $P\sqrt{2}$ acting along BD into two components, $P\sqrt{2}$ cos 45° i.e, P olong BA and $P\sqrt{2}$ sing 45° i.e. P along BC. The two forces $P, -P$ acting along CD will balance each other and the two forces P, P acting along AB and BA balance each other. The three forces $2P$, P and P acting along BC, BC and CB respectively will have a resultant $2P$ along BC. Hence, the given system of forces is equivalent to two equal and parallel forces $2P$, $2P$ acting along BC and DA. Hence, they form a couple of moment $2aP$. Hence the result.

Ex. 2. Three forces acting along the sides of a triangle, taken in order, are equivalent to a couple. Show that they are proportional to the sides of the triangle.

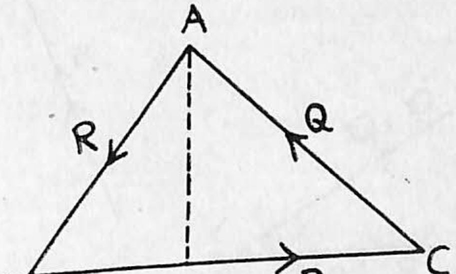


FiG. 81

Let the forces P, Q and R act along BC, CA and AB. Let $AD \perp BC$. Let a, b, c be the lengths of the sides BC, CA and AB respectively of $\triangle ABC$ and let $\triangle$ be the area of $\triangle ABC$. Then $AD \times a = 2\triangle$.

$\therefore AD = \frac{2\triangle}{a}$. Since the forces are equivalent to a couple, the algebraic sum of the moments of the forces about any point in their plane will be the same. Hence, the algebraic sum of the moments of P, Q and R about A, B and C must be equal. Now, the algebraic sum of the moments of P, Q and R about A is $P.\ AD = P.\ \frac{2\triangle}{a}$. Similarly, the algebraic sums of the moments of the forces about B and C are respectively

$$Q.\frac{2\triangle}{b} \text{ and } R.\frac{2\triangle}{c}.$$

$$\text{Hence, } p.\frac{2\triangle}{a} = Q.\frac{2\triangle}{b} = R.\frac{2\triangle}{c}$$

$$= \frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$$

i.e. the forces are proportional to the sides of the triangle ABC.

Ex. 3. Two systems of three forces P, Q, R and P', Q', R' act along the sides of a triangle, taken in order. If they produce parallel resultants, prove that

$$\begin{vmatrix} P & Q & R \\ P' & Q' & R' \\ p & q & r \end{vmatrix} = 0,$$

where p, q, r are a system of concurrent forces which, acting paralell to the sides of the triangle, will be in equilibrium.

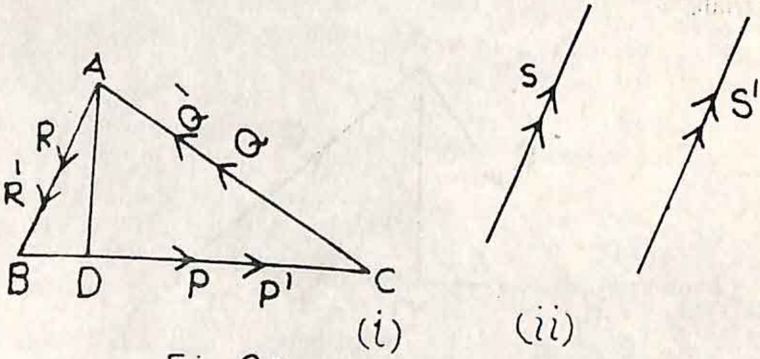


FIG. 82

Let the forces P, Q, R and P', Q', R' act along BC, CA and AB respectively of a triangle ABC. Let S be the resultant of P, Q and R and S' be that of P', Q' and R'. Since these two resultants are given to be parallel, we can write

$\therefore S' = KS$, where K is a constant.

Since S is the resultant of P, Q and R, hence KS is the resultant of PK, QK and RK which will act along BC, CA and AB of $\triangle$ ABC. Now, S' and $-KS$ being unlike, equal and parallel forces, will form a couple. Hence, P', Q', R' and $-KP, -KQ, -KR$ acting along the sides BC, CA and AB of $\triangle$ ABC will form a couple.

In other words, the forces $P'-KP, Q'-KQ$ and $R'-KR$ acting along BC, CA and AB of $\triangle$ ABC will form a couple. Hence, the algebraic sums of the moments of these forces about A, B and C will be equal. If $AD \perp BC$, a, b, c be the lengths of the sides BC, CA and AB of $\triangle$ ABC and $\triangle$ is the area of $\triangle$ ABC, then

$$AD \times a = 2\triangle$$

$$\Rightarrow AD = \frac{2\triangle}{a}.$$

The algebraic sum of the moments of the forces $P'-KP$, $Q'-KQ$ and $R'-KR$ about A is

$$(P'-KP)\ AD = (P'-KP) \times \frac{2\triangle}{a}.$$

Similarly, the algebraic sums of the moments of these forces about B and C are respectively

$$(Q'-KQ)\ \frac{2\triangle}{b} \text{ and } (R'-KR)\ \frac{2\triangle}{c}.$$

Now, since the forces p, q, r act at a point, are parallel to the sides of $\triangle$ ABC and are in equilibrium, they must be proportional to the sides of the triangle ABC i.e.

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = K', \text{ say.}$$

$$\therefore a = \frac{p}{K'},\ b = \frac{q}{K},\ c = \frac{r}{K'}.$$

Hence, the algebraic sum of the moments of the forces about A becomes $(P'-KP)\dfrac{2\triangle K'}{p}$,

that about B is

$$(Q'-KQ)\frac{2\triangle K}{q}$$

and about C is

$$(R'-RK)\ \frac{2\triangle K'}{r}.$$

Each of these moments must be equal to the moment m of the couple formed by these three forces. Hence,

$$(P'-KP)\,\frac{2\triangle K'}{\mathrm{p}} = \mathrm{m}$$

$$\Rightarrow P'-KP-\lambda\,\mathrm{p} = 0, \text{ where } \lambda = \frac{\mathrm{m}}{2\triangle K'}.$$

Similarly, $Q' - KQ - \lambda\,\mathrm{q} = 0$
and $R' - KR - \lambda\,\mathrm{r} = 0.$

Eliminating K and λ, we get

$$\begin{vmatrix} P' & P & \mathrm{p} \\ Q' & Q & \mathrm{q} \\ R' & R & \mathrm{r} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} P & Q & R \\ P' & Q' & R' \\ \mathrm{p} & \mathrm{q} & \mathrm{r} \end{vmatrix} = 0.$$

Hence the result.

Ex. 4. A system of forces in the plane of rectangular plate $ABCD$ (AB = a, BC = b) have the moments G_1, G_2 and G_3 about the points A, B, C respectively. Show that the magnitude of the resultant is :

$$R = \sqrt{\left(\frac{G_1-G_2}{\mathrm{a}}\right)^2 + \left(\frac{G_3-G_2}{\mathrm{b}}\right)^2}$$

and find the distance from A of the point where its line of action intersects AB. Discuss the case $G_1 = G_2 = G_3$.

Since the moment of the system of forces about A and B are not zeroes, the resultant of the forces does not pass through A and B. Let the resultant R cut AB at E, where AE=x.

Let X and Y be the components of R along AB and perpendicular to it through E. Now, taking moments of the forces X and Y about A, we get

$$\mathrm{x}Y = G_1$$

Taking moments about B, we get ...(1)

$$(\mathrm{a}-\mathrm{x})\ \mathrm{y} = -\,G_2$$

Taking moments about C, we get ...(2)

$$\mathrm{b}X-(\mathrm{a}-\mathrm{x})\ \mathrm{y} = G_3$$

...(3)

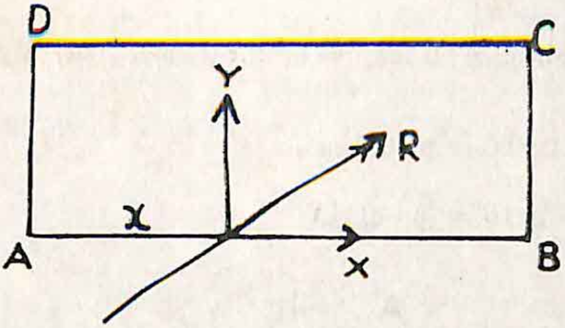


FiG. 83

From (1), $y = \frac{G_1}{x}$ and from (2), $y = \frac{G_2}{x-a}$.

Hence, $\frac{G_1}{x} = \frac{G_2}{x-a}$

$$\Rightarrow G_1(x-a) = G_2 x$$
$$\Rightarrow x(G_1 - G_2) = aG_1$$
$$\Rightarrow x = \frac{aG_1}{G_1 - G_2}.$$

Hence, the line of action of the resultant cuts *AB* at a distance $\frac{aG_1}{G_1 - G_2}$ from *A*.

From (2) and (3), we get

$$b\,X + G_2 = G_3 \Rightarrow X = \frac{G_3 - G_2}{b}.$$

Also, $Y = \frac{G_1}{x} = \frac{G_1(G_1 - G_2)}{aG_1} = \frac{G_1 - G_2}{a}$

and $R = \sqrt{X^2 + Y^2}$

$$= \sqrt{\left(\frac{G_1 - G_2}{a}\right)^2 + \left(\frac{G_3 - G_2}{b}\right)^2}.$$

If $G_1 = G_2 = G_3$, then $R = 0$, but $G_1 \neq 0$.

$\therefore$ The system of forces reduces to a couple.

Ex. 5. *X*, *Y*, *Z* are the points on the sides *BC*, *CA* and *AB* of $\triangle ABC$ such that $BX{:}XC = CY{:}YA = AZ{:}ZB = p{:}q$. Prove that the forces represented by *AX*, *BY*, *CZ* reduce to a couple of moment

$\frac{2\triangle(p-q)}{p+q}$, where $\triangle$ is the area of the triangle ABC. We join AX, BY and CZ.

Since $BX : XC = p : q$, hence

$$p\,\overrightarrow{AC} + q\,\overrightarrow{AB} = (p+q)\,\overrightarrow{AX} \quad \text{..(1)}$$

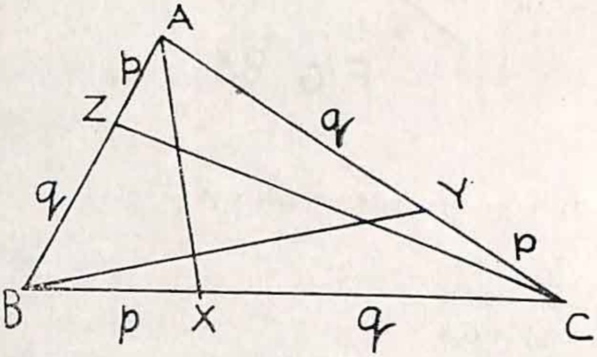


FIG. 84

Similarly,

$$p\,\overrightarrow{BA} + q\,\overrightarrow{BC} = (p+q)\,\overrightarrow{BY} \quad \text{...(2)}$$

$$\text{and } p\,\overrightarrow{CB} + q\,\overrightarrow{CA} = (p+q)\,\overrightarrow{CZ} \quad \text{...(3)}$$

Adding (1), (2) and (3), we get

$$p\,\overrightarrow{BA} - q\,\overrightarrow{BA} + p\,\overrightarrow{AC} - q\,\overrightarrow{AC} + p\,\overrightarrow{CB} - q\,\overrightarrow{CB}$$

$$= (p+q)(\overrightarrow{AX} + \overrightarrow{BY} + \overrightarrow{CZ})$$

$$\Rightarrow (p-q)(\overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CB}) = (p+q)(\overrightarrow{AX} + \overrightarrow{BY} + \overrightarrow{CZ}).$$

But $\overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CB}$ is equal to a couple of moment $2\triangle$. Hence,

$\overrightarrow{AX} + \overrightarrow{BY} + \overrightarrow{CZ}$ is equivalent to a couple of moment $\frac{2\triangle(p-q)}{p+q}$.

Hence the result.

Ex. 6. Three parallel forces P, Q and R acting at the angular points of a triangle ABC are in equilibrium when they are perpendicular to the side BC. If their lines of action are turned through a given angle in the same sense, show that they are equivalent to a couple.

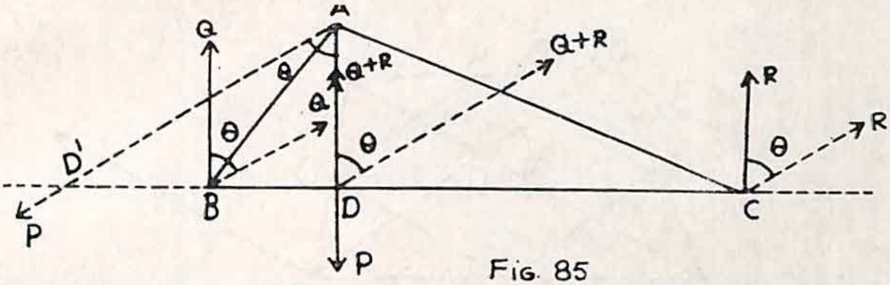


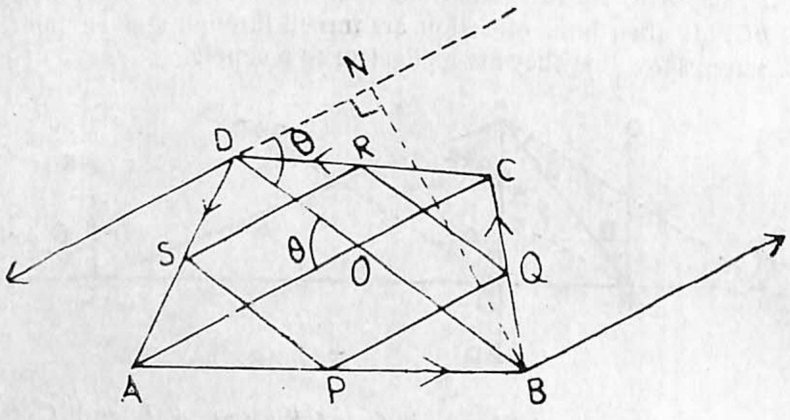
Fig. 85

Let the three parallel forces P, Q and R act at A, B and C respectively in the directions as shown in the figure. Let the line of action of the force P cut BC at D. When the lines of action of these forces are perpendicular to BC, then the forces are given to be in equilibrium. Hence, the direction of one of the forces, say, P will be opposite to those of the other two.

It P acts downwards and Q and R act upwards. then $P=Q+R$ and the resultant force $Q+R$ of like parallel forces Q and R must be along DA for equilibrium. Let all the forces be now rotated through an angle θ in the clockwise directions (shown by dotted lines in the figure). Then the resultant of Q and R is now a force $Q+R$ acting upwards at D making an angle of $90°-\theta$ with BC and the force P will now cut CB produced at D' such that it is parallel to $Q+R$ and at a distance $AD \sin \theta$ from the line of action of the force $Q+R$. Since the forces P and $Q+R$ are equal, unlike and parallel, they form a couple.

Ex. 7. Show that the forces represented by the sides AB, BC, CD, DA of a quadrilateral can not be in equilibrium but are equivalent to a couple whose moment is equal to four times the area of the parallelogram whose vertices are the middle points of the sides.

Let P, Q, R and S be the middle points of the sides AB, BC, CD and DA respectively. We join AC and BD. Now, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ and the resultant of the forces $\overrightarrow{AB}$ and $\overrightarrow{BC}$ will act at B parallel to AC. Similarly, $\overrightarrow{CD}+\overrightarrow{DA}=\overrightarrow{CA}$ and hence the resultant of the force CD and DA will act at D parallel to CA.



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These two equal, unlike and parallel forces will from a couple of moment $AC \times BN$, where BN is the perpendicular drawn form B to the line of action of the resultant force at D.

Let the diagonals AC and BD of the quadrilateral $ABCD$ intersect at O and let $\angle AOD=\theta$. Now, since P, Q, R and S are the middle points of AB, BC, CD and DA respectively,

$$PQ = SR = \frac{1}{2}\ AC,\ PQ \parallel AC \text{ and } SR \parallel AC$$

$$\text{and } SP=QR = \frac{1}{2}\ BD,\ SP \parallel BD \text{ and } QR \parallel BD.$$

Since $AC \parallel DN$, $\therefore\ \angle BDN = \angle AOD = \theta$

$\therefore\ BN = BD \sin\theta = 2\,SP \sin\theta$

and $AC = 2\,PQ$.

Hence, the moment of the couple is

$$\begin{aligned} AC \times BN &= 2\,PQ.\,2SP \sin\theta \\ &= 4\,PQ \times SP \sin\theta \\ &= 4\,PQ \times SP \sin(180^\circ - \angle SPQ) \\ &= 4\,PQ \times SP \sin\angle SPQ \\ &= 4\times \text{ area of the parallelogram } PQRS. \end{aligned}$$

Hence the result.

Ex. 8. $ABCD$ is a rectangle such that $AB=CD=$a and $BC=DA=$b. Forces P act along AD and CB and forces Q act along AB and CD.

Prove that ihat the perpendicular distance between the resultant of the forces P, Q at A and the resultant of the forces P, Q at C is

$$\frac{Pa-Qb}{\sqrt{P^2+Q^2}}.$$

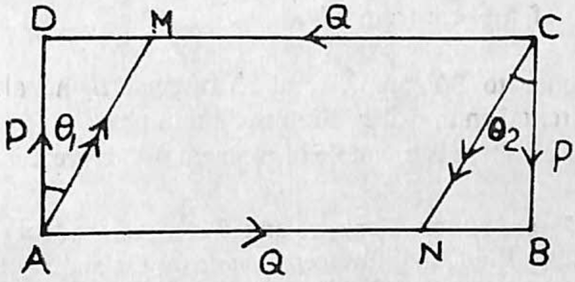


FIG. 87

Let the resultant of P and Q at A act along AM and that of P and Q at C act along CN. The magnitude of the resultant at each of the points A and C is $\sqrt{P^2+Q^2}$.

If $\angle MAD = \theta_1$, then

$$\tan\theta_1 = \frac{Q\sin 90^\circ}{P+Q\cos 90^\circ} = \frac{Q}{P}.$$

Similarly, if $\angle BCN = \theta_2$, then

$$\tan\theta_2 = \frac{Q\sin 90^\circ}{P+Q\cos 90^\circ} = \frac{Q}{P}$$

Hence $\theta_1 = \theta_2$ and so $AM \parallel CN$.

Hence, these forces form a couple of moment

$d\sqrt{P^4+Q^2}$, where d is the distance between these two resultant forces. Originally, we had two couples (Q, AD) and (P, AB). The algebraic sum of the moments of these two couples $= Pa - Qb =$ the moment of the resultant couple.

$$\therefore\ d\sqrt{P^2+Q^2} = Pa - Qb$$

$$\therefore\ d = \frac{Pa-Qb}{\sqrt{P^2+Q^2}}$$

EXERCISE 6.1

1. Forces P, $5P$, $9P$ and $5P$ act along the sides AB, BC, CD and DA respectively of a rectangle $ABCD$ where $AB = 8$ cm and $BC = 6$ cm. Find the magnitude and line of action of the resultant.

   [Ans. $8P$. The line of action of the resultant is parallel to CD and at a distance of $5\frac{3}{4}$ cm from C] .

2. Forces equal to $3N$, $5N$, $3N$ and $5N$ respectively act along the sides of a square, taken in order. Find their resultant.

   [Ans. The resultant is a couple of moment 8a, where a is a side of the square.]

3. Ox and Oy are rectangular axes and P is a point whose co-ordinates are (5, 12). Find the intercepts made on Ox and Oy by the line of action of the resultant of a force of 13 units along OP and a couple of moment 65 units in the sense from Ox to Oy.

   $\left[\text{Ans. } 5\frac{5}{12}, -13\right]$

4. Forces of magnitudes 1,2,3,4 $2\sqrt{2}$ act respectively along the sides AB, BC, CD, DA and the diagonal AC of the square $ABCD$. Show that their resultant is a couple and find its moment. [*C.U.*' 47]

   [Ans. 5a, where a is the side of the square]

5. $ABCDEF$ is a regular hexagon. Forces of magnitudes P, $2P$, $3P$, $5P$ and $6P$ act along AB, BC, CD, EF and AF respectively. Show that a force can be determined to act along ED so that the six forces are equivalent to a couple.

6. ABC is an equilateral triangle of side a. D, E and F divide the sides BC, CA and AB respectively in the ratio 2:1. Three forces each equal to P act at D, E and F perpendicularly to the sides and directed away from the triangle. Prove that they are equivalent to a couple of moment $\frac{1}{2}$ Pa.

7. Four forces are completely represented by the sides AB, AD, CB and DC of a quadrilateral $ABCD$. Show that they are equivalent to a couple consisting of two forces through A and C, each equal and parallel to other diagonal BD.

8. Forces $3P$, $4P$ and $5P$ respectively act along the sides of a right-angled triangle of sides 3, 4 and 5 in order. Find the forces acting

at the ends of the greatest side which will equilibriate the system.

[Ans. Two forces each $\frac{12}{5}P$ forming a couple of moment $12P$]

9. Unlike parallel forces each equal to $4N$ act along a pair of opposite sides of length 2m of a rectangle. Find the magnitude of the forces which acting along the other sides of length $\frac{1}{2}$ m will form with these a system in equilirbium.

[Ans. 1 N]

10. Four forces are completely represented by the sides BA, AD, DC and CB of a quadrilateral $ABCD$. Show that they are equivalent to a couple consisting of the two forces through A and C each equal and parallel to the diagonal BD.

11. Two couples with forces acting along the sides of a parallelogram are in equiliribum. Find the ratio of the forces of the couples.

[Ans. These are proportional to the sides of the parallelogram]

12. Four forces acting along the sides of a parallelogram are equivalent to a couple. Show that the forces along the opposite sides are equal in magnitude and opposite in sense.

13. $ABCDEF$ is a regular hexagon. If three forces are represented in all respects by AC, CE and EA, show that they are equivalent to a couple whose moment is equal to the area of the hexagon.

14. Forces of $1N$, $2N$, $3N$, $4N$, $5N$ and $6N$ act along the sides AB, BC, CD, DE, EF and FA of a regular hexagon, taken in order. Find the single force acting at A and the moment of the couple which together are equivalent to the system.

[Ans. $6N$ along DA produced, $\frac{21}{2}\sqrt{3}$ times AB]

15. If the algebraic sum of the moments of two given coplanar forces about any point in their plane is a non-zero constant, show that the given forces form a couple.

16. Three forces proportional to the sides of a trinagle act perpendicular to these sides, all inwards. Show that either they are in equilibrium or they form a couple.

17. Forces of magnitudes 1,2,3,4,6,5 and 4 units act respectively along the sides AB, CB, CD, ED, EF and AF of a regular hexagon, the sense of each force being indicated by the order of the letters. Prove that the six forces are equivalent to a couple of moment $\frac{3}{2}\sqrt{2}$ a, where a is the side of the hexagon.

18. *ABCDEF* is a regular hexagon. Forces P, $2P$, $3P$, $5P$ and $6P$ act along *AB*, *BC*, *DC*, *EF* and *AF* respectively. Show that a force can be determined to act along *ED* so that the six forces are equivalent to a couple and find its moment.

[Ans. $-3\sqrt{3}\,aP$ where 2a is the side of the hexagon]

19. *ABCD* is a square whose side is 2 units in length. Forces a,b,c,d act along the sides *AB*, *BC*, *CD*, *DA*, taken in order, and forces $p\sqrt{2}$, $q\sqrt{2}$ along *AC* and *DB* respectively. Show that if $p+q=c-a$ and $p-q=d-b$, the forces are equivalent to a couple of moment $a+b+c+d$.

20. *ABCD* is a rectangle in which $AB=4a$ and $BC=3a$. Forces of magnitudes P, $2P$, $3P$, $4P$ and $5P$ act respectively along *AB*, *BC*, *CD*, *AD* and *DB*. Reduce the set of forces to
    (i) a force at *A* and a couple
    (ii) a force at *B* and a couple.

[Ans. (i) $\sqrt{13}\,P$, $5aP$ (ii) $\sqrt{13}\,P$, $-7aP$]

21. Prove that the forces represented in magnitude and line of action by the sides of two triangles, taken opposite ways round, are in equilibrium, provided the triangles are of equal area.

22. *D*, *E* and *F* divide the sides *BC*, *CA* and *AB* respectively of an equilateral triangle *ABC* of side a in the ratio 5:1. Three forces each equal to P act at *D*, *E* and *F* perpendicular to the sides and outward from the triangle. Show that they are equivalent to a couple of moment Pa.

[*C.U.*' 43]

23. Show that the forces 1,5,3,4,2 and 6 units acting in order round the sides of a regular hexagon, which touch a circle of radius a, are equivalent to a couple. Find its moment and show that if the forces 2 and 5 units be replaced by forces $8\frac{1}{2}$ and $5\frac{1}{2}$ units acting along the same sides but in directions opposite to the forces 2 and 5 respectively, the system will be in equilibrium.

[Ans. 21a]

24. If three forces P, Q and R acting at the angular points of a triangle *ABC* along the tangents to the circum-circle are equivalent to a couple, show that

$P : Q : R = \sin 2A : \sin 2B : \sin 2C$.

25. Two unlike parallel forces P, Q act on a body. If two equal and opposite forces S in any two parallel lines at a distance b appart are combined with them, show that the resultant is displaced by a distance $\dfrac{bS}{P-Q}$

26. X, Y and Z are points on the sides BC, CA and AB of $\triangle ABC$ such that $\frac{BX}{XC}=\frac{CY}{YA}=\frac{AZ}{ZB}=\frac{3}{1}$. Three forces are represented in magnitude and direction by the sides AX, BY and CZ. Show that they are equivalent to a couple whose moment is equal to the area of $\triangle ABC$.

27. P and Q are two like parallel forces. If a couple each of whose forces is F, and whose arm is a, in the plane of P and Q, is combined with them, show that the resultant is displaced through a distance $\frac{Fa}{P+Q}$.

28. A system of forces in the plane of a square plate $ABCD$ has the moments G_1, G_2 and G_3 about A, B and C respectively. Show that the moment of the system about D is equal to $G_1 + G_3 - G_2$.

29. The constituents of a couple G act at A and B. If their lines of action are turned through a right angle in the same sense, they will form a couple of moment H. When they both act at right angles to AB, show that they form a couple of moment $\sqrt{G^2 + H^2}$.

30. Forces λa, λb, λc parallel to the sides of a triangle act at A', B', C', the centres of the escribed circles. Show that they are equivalent to a couple of moment 2λ (a+b+c) R, where R is the radius of the circumscribed circle.

31. $ABCD$ and $A'\,B'\,C'\,D'$ are any two coplanar parallelograms. If forces act along AA', $B'B$, CC' and $D'D$ represented by these respective lengths, show that they reduce to a couple.

32. Three forces λa, λb and λc act along AO, BO and CO respectively, where O is the ortho-centre of $\triangle ABC$. If they are rotated through the same angle θ about A, B and C respectively, show that they are equivalent to a couple of moment $4\lambda \triangle \sin\theta$, where $\triangle$ is the area of $\triangle ABC$.

33. If the forces completely represented by the sides of a triangle, taken in order, are in equilibrium with three equal forces acting at the vertices of the triangle along the tangents to the circumcircle, the same way round, prove that the triangle must be equilateral.

34. A uniform bar of weight W and length 2a is suspended by two equal strings of length l from two points in the same horizontal plane and at a distance 2b apart. Show that the moment of the couple

required to keep the bar twisted through an angle θ about the vertical axis is

$$\frac{Wa^2 \sin\theta}{\left(l^2 - 4a^2 \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

35. A circular disc of weight 4 units and radius $1\frac{1}{2}$ unit is suspendeds horizontally by four equal vertical strings of length 3 units, attached symmetrically to points on its circumference. What is the magnitude of the horizontal couple required to keep the disc twisted through an angle of 60° ?

[Ans. 3 units]

Chapter VII

Resultant of Coplanar Forces acting on a Rigid Body and General Conditions of Equilibrium

7.1 Reduction of a system of coplanar forces acting on a rigid body

Theorem 1: A system of coplanar forces acting on a rigid body can be reduced to a single force acting at any arbitrary point in their plane together with a couple in the plane.

Let P_1, P_2, P_3,. P_n be the system of forces acting at the points $A_1 : (x_1, y_1)$, $A_2 : (x_2, y_2)$,......$A_n : (x_n, y_n)$ respectively with reference to two perpendicular lines Ox and Oy through an arbitrarily chosen point O in the plane of the forces. Let the directions of P_1, P_2,.... P_n make angles $\alpha_1, \alpha_2, \ldots \alpha_n$ respectively with Ox. If X_r, Y_r be the components of P_r $(r=1, 2, \ldots n)$ along Ox and Oy respectively, then

$X_r = P_r \cos \alpha_r,$

$Y_r = P_r \sin \alpha_r \; (r=1, 2, \ldots\ldots n).$

We now introduce at O a pair of equal and opposite forces X_1, X_1 acting along Ox and Ox' and a pair of equal and opposite forces Y_1, Y_1 acting along Oy and Oy'. These forces which balance one another will have no effect on the given system of forces. Now, the components X_1 at A_1 and X_1 at O in the direction Ox' will form a couple of moment $-y_1 X_1$. Similarly, the components Y_1 at A_1 and Y_1 at O along Oy' form a couple of moment $x_1 Y_1$. The forces which are left now are X_1 along Ox and Y_1 along Oy at O. Thus the force P_1 at A_1 is equivalent to the components X_1 and Y_1 along Ox and Oy respectively together with a couple of moment equal to the algebraic sum of these two moments i.e. $(x_1 Y_1 - y_1 X_1)$. Similarly, the force P_2 at A_2 can be replaced by the components X_2 and Y_2 along Ox and Oy respectively at O together with a couple of moment $x_2 Y_2 - y_2 X_2$ and similarly for every force of the system. Combining all the components along Ox and Oy separately and all the couples, we can reduce the given system of forces as

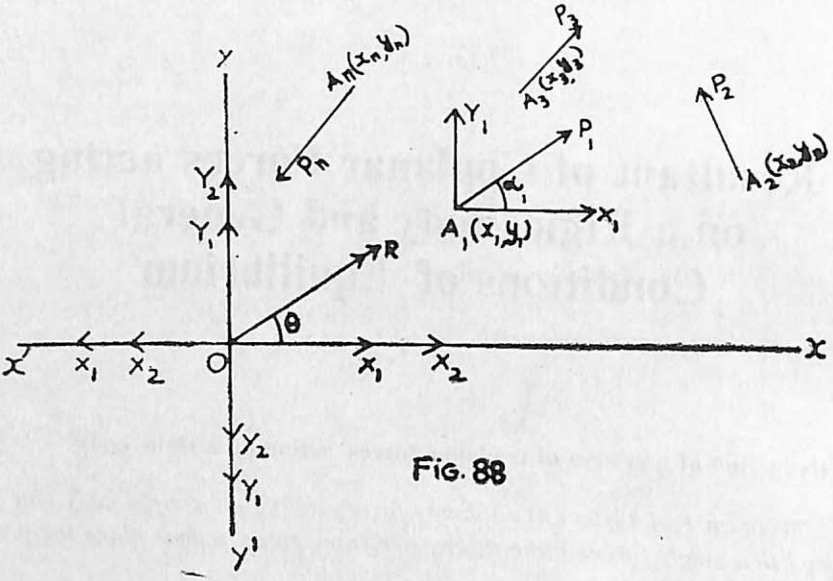


FIG. 88

(i) a component force $= \sum\limits_{r=1}^{n} X_r = \sum\limits_{r=1}^{n} P_r \cos \alpha_r$ along Ox

(ii) a component force $= \sum\limits_{r=1}^{n} Y_r = \sum\limits_{r=1}^{n} P_r \sin \alpha_r$ along Oy

and (iii) a single couple of moment G given by

$$G = \sum_{r=1}^{n} (x_r Y_r - y_r X_r)$$

$$= \sum_{r=1}^{n} (x_r P_r \cos \alpha_r - y_r P_r \sin \alpha_r). \qquad \ldots(1)$$

The two component forces along Ox and Oy can be combined to give a single resultant force R acting at O in a direction making an angle θ with Ox such that

$$R = \sqrt{\left(\sum_{r=1}^{n} X_r\right)^2 + \left(\sum_{r=1}^{n} Y_r\right)^2} \qquad \ldots(2)$$

and

$$\left.\begin{aligned} R\cos\theta &= \sum_{r=1}^{n} X \\ R\sin\theta &= \sum_{r=1}^{n} Y_r\,. \end{aligned}\right\} \qquad \ldots(3)$$

Thus the given system of forces reduces to a single force R at an arbitrary point O together with a couple of moment G.

We also observe that $R\cos\theta = \Sigma X_r$ and $R\sin\theta = \Sigma Y_r$, so that the algebraic sums of the resolved parts of the given forces along any two mutually perpendicular directions are respectively equal to the resolved parts of the single resultant in the same direction. Also from (1), we see that the moment of the couple is equal to the algebraic sum of the moments of the given forces about the arbitrarily chosen point O.

Remark: For different choices of the point O, the single force R will always be the same in magnitude and direction, for its resolved part in any direction is always equal to the sum of the resolved parts of the given forces. The moment of the couple will, however, be different for different choices of the point O.

Theorem 2: A system of coplanar forces acting on a rigid body can be reduced either to a single force or to a single couple unless it is in equilibrium.

By theorem 1, the given system of coplanar forces can be reduced to a single force R acting at any arbitrary point together with a couple of moment G in the plane. There may be several cases to be considered.

(i) Let $R \neq 0$ and $G \neq 0$.

In this case, the resultant of the force R and the couple G is a force R acting parallel to the original direction at a distance $\frac{G}{R}$ from its original line of action.

(ii) Let $G=0$ but $R \neq 0$.

In this case, the system of forces clearly reduces to the force R at the arbitrary point O.

(iii) Let $R=0$ but $G \neq 0$.

In this case, the given system of forces reduces to a couple only. The case $R=0$ and $G=0$ is not to be considered, since the given system of forces is not in equilibrium.

Hence, we see that if the system of coplanar forces be not in equilibrium, the system reduces to a single force or a single couple in all cases.

Remark: We have seen that the algebraic sum of the moments of a systme of coplanar forces about the origin is given by

$$G=\sum_{r=1}^{n} (x_r\, Y_r-y_r\, X_r).$$

If we need to find the algebraic sum of the moments of the same system of forces about any other point, say (h, k), then this moment G' is given by

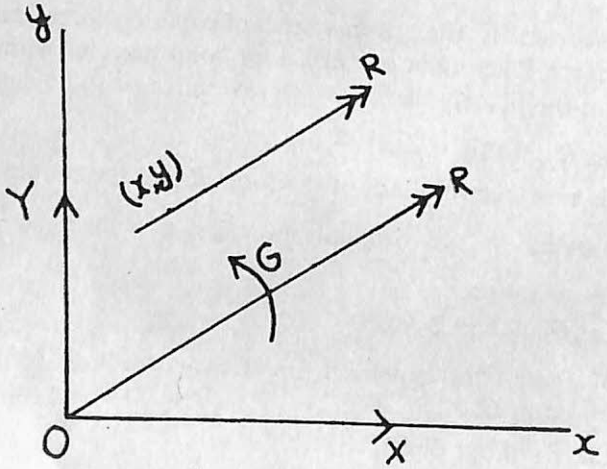
$$G'=\sum_{r=1}^{n} (x_r-h)\, Y_r-(y_r-k)\, X_r=G-h\,\Sigma Y_r+k\,\Sigma X_r$$

$$=G-hY+kX,$$

where, $X=\Sigma X_r$ and $Y=\Sigma Y_r$.

7.2 Equation of the line of action of the resultant of coplanar forces acting on a rigid body

We have seen in section 7.1 that a system of coplanar forces acting on a rigid body can be reduced to a single force R with components



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$$X=\sum_{r=1}^{n} X_r \text{ along x-axis and } Y=\sum_{r=1}^{n} Y_r \text{ along y-axis}$$

together with a couple of moment G given by

$$G=\sum_{r=1}^{n}(x_r Y_r-y_r X_r).$$

Now, this single force R and the couple G can again be combined into a single resultant force whose magnitude and direction will be the same as those of R at O. To get the position of the line of action of this single resultant R, we assume that (x, y) are the co-ordinates of any point on its line of action. Hence, the algebraic sum of the moments of the given forces X along Ox and Y along Oy about the point (x, y) together with the couple of moment G must be zero. Hence, taking moments about (x, y), we get $yX-xY+G=0$, which is the required equation of the line of action of the resultant.

7.3 Conditions for a single resultant force or a single couple

We have seen that a system of coplanar forces acting on a rigid body is equivalent to a single force R acting at an arbitrary point O with components X and Y along two mutually perpendicular lines Ox and Oy respectively together with a couple of moment G which is equal to the algebraic sum of the moments of all the forces about **O.** Since $R^2= X^2+Y^2$, it is clear that $R=0$ if and only if both $X=0$ and $Y=0$. Thus we get the following results :

(i) *A system of coplanar forces acting on a rigid body will reduce to a single force only if* $G=0$ *i.e. if the algebraic sum of the moments of the forces about any point in their plane is zero, but the algebraic sum of the resolved parts in at least one of two mutually perpendicular directions through any arbitrary point in their plane is not equal to zero.*

(ii) *The above system of forces will reduce to a single couple only if the algebraic sum of the resolved parts of the forces in each of two mutually perpendicular directions through any arbitrary point in their plane is separately zero.*

Since the moment of a couple which is equal to the algebraic sum of the moments of two forces, forming the couple, about any point in their plane is a constant, we can also say that *the system of forces will reduce to* *a couple if the algebraic sum of the moments of all the forces about any point in their plane is a non-zero constant.*

7.4 General conditions of equilibrium of a system of coplanar forces acting on a rigid body

Theorem 1: The necessary and sufficient conditions that a system of coplanar forces acting on a rigid body may be in equilibrium are that

(*i*) *the algebraic sum of the resolved parts of the forces in any two mutually perpendicular directions should be separately zero and*

(*ii*) *the algebraic sum of the moments of the forces about any point in their plane should also be zero.*

Let the system of coplanar forces be reduced to a single resultant force R acting at any arbitrary point O and a couple of moment G which is equal to the algebraic sum of the moments of the forces about O. Let X and Y be the components of R along two mutually perpendicular lines Ox and Oy through O respectively so that X and Y are the algebraic sums of the resolved parts of the forces along OX and Oy respectively.

$\therefore R^2=X^2+Y^2$.

To prove that the conditions are necessary for equilibrium, we assume that the system of forces is in equilibrium. Now, since a couple and a force cannot maintain equilibrium, hence, for equilibrium, both R and G should be zero. Now, $R=0 \Rightarrow X=0$ and $Y=0$.

Hence, $X=Y=G=0$, which proves that the force system being in equilibrium, the conditions $X=Y=G=0$ are necessary.

Now, to prove that the conditions $X=Y=G=0$ are sufficient for equilibrium, let the system of forces be such that the conditions $X=Y=G=0$ are satisfied. We shall prove that the system of forces is in equilibrium. We show that if any one of X, Y and G is non-zero, then the equilibrium is not possible.

Let $X=0$, $Y=0$ but $G\neq 0$.

In this case, the forces will be reduced to a couple and so the equilibrium is not possible.

Let $X=0$, $G=0$, but $Y\neq 0$; or $Y=0$, $G=0$, but $X\neq 0$. In either of the cases, the system will be reduced to a single force and so the equilibrium is not possible. Hence, the only case for equilibrium is $X=Y=G=0$.

Hence, the given conditions are both necessary and sufficient.

Another set of necessary and sufficient conditions of equilibrium of a given system of coplanar forces acting on a rigid body is given by the following theorem :

Theorem 2: A system of coplanar forces acting on a rigid body is in equilibrium if the algebraic sum of their moments about any three non-collinear points in their plane is separately zero.

Let the three points in the plane of the forces be the origin O, the points A_1: (x_1, y_1) and A_2: (x_2, y_2), where O, A_1 and A_2 are not collinear. Let the algebraic sums of the moments of the forces about O, A_1 and A_2 be G, G_1 and G_2 respectively. If X and Y be the algebraic sums of the resolved parts of the forces along Ox and Oy, the two mutually perpendicular lines respectively, then taking moments about A_1 and A_2, we get

$$G_1 = G - x_1\, Y + X$$

and $G_2 = G - x_2\, Y + y_2\, X$.

But it is given that $G = G_1 = G_2 = 0$.

Hence, we have

$$x_1\, Y - y_1\, X = 0 \qquad \ldots(1)$$

and $x_2\, Y - y_2\, X = 0. \qquad \ldots(2)$

Multiplying (1) by x_2 and (2) by x_1 and subtracting, we get

$$(x_1\, y_2 - x_2\, y_1)\, X = 0. \qquad \ldots(3)$$

Multiplying (1) by y_2 and (2) by y_1 and subtracting, we get

$$(x_1\, y_2 - x_2\, y_1)\, Y = 0. \qquad \ldots(4)$$

Now, since the points $(0, 0)$, (x_1, y_1) and (x_2, y_2) are not collinear, hence the slope of the line joining $(0, 0)$ and (x_1, y_1) and that of the line joining (x_1, y_1) and (x_2, y_2) will be unequal.

Hence, $\dfrac{y_1}{x_1} \neq \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow y_1\, x_2 \neq x_1\, y_2$.

Hence, $x_1\, y_2 - x_2\, y_1 \neq 0$.

The equations (3) and (4) are, therefore, true only when

$$X = Y = 0.$$

Also it is given that $G = 0$.

Hence, by theorem 1, the system of forces is in equilibrium.

There is still a third form of the conditions of equilibrium of a system of forces which can be stated as follows:

Theorem 3 : A system of coplanar forces acting on a rigid body is in equilibrium if the algebraic sum of the moments of the forces about each *of two different points is separately zero, and the algebraic sum of the resolved parts of the forces in a direction, not perpendicular to the join of the two points is also zero.*

Let the two points be the origin O and A_1: (x_1, y_1). Let G and G_1 be the algebraic sum of the moments of the forces about O and A_1 respectively. Let the given direction be that of Ox. According to the given condition $x_1 \neq 0$.

Taking moment about A_1, we have $G_1 = G - x_1 Y + y_1 X$, where X and Y have the same meaning as stated in theorem 2.

It is given that $G_1 = G = X = 0$.

Hence, it follows that $x_1 \; Y = 0$.

But since $x_1 \neq 0$, Y must be zero.

Thus we have $X = Y = G = 0$.

Hence, the system of forces is in equilibrium.

Remark 1: We have seen that for equilibrium of a *particle* acted upon by a number of forces, it is both necessary and sufficient that $X=0$ and $Y=0$ and no other conditions are needed, whereas in the case of equilibrium of a *rigid body* acted upon by a number of coplanar forces, only these two conditions cannot be both necessary and sufficient. This is because the only motion of a particle is its translation which can be stopped by putting $X=0$ and $Y=0$. But a rigid body possesses not only a motion of translation, but also a motion of rotation and this latter motion can be stopped by imposing the third condition viz. $G=0$, since a couple causes the motion of rotation of a body.

Remark 2: One of the conditions of equilibrium as stated in Theorem 1 is that the algebraic sum of the resolved parts of the forces in any two mutually perpendicular directions should be separately zero. The theorem is still true if we take any two different directions in the plane instead of two mutually perpendicular directions.

7.5 Equilibrium of a constrained body

If one point of a rigid body is fixed, the body can turn about that point without having any motion of translation. If two points of a body are fixed, then the body can turn about the line joining the two points. In this case also, the body can not have any translatory motion. If three non-collinear points of a body are fixed, then the body cannot have any rotatory or translatory motion and so the body cannot move at all. If one or more points of a body are fixed, then such a body is called a *constrained body*. When a point in a body acted upon by external forces becomes fixed, an unknown force passes through this point. Such an unknown force is called *the force of constraint*. Suppose that one point O of a rigid body under the action of a system of forces in a plane through O is fixed and let the body be in equilibrium. The system of coplanar forces can be reduced to a single force R acting at O together with a couple of moment G which is equal to the algebraic sum of the moments of all the forces about O. Let F be the force of constraint at O. Hence, the body is in equilibrium under the forces R, F and the couple G. Hence, R and F acting at O must be equal and opposite and G must be zero.

Thus *when a constrained body has only one point fixed and is acted upon by a system of forces in a plane passing through the point, it will be in equilibrium if the algebraic sum of the moments of the forces about the fixed point is zero.*

Now, let the body instead of having one point fixed have two fixed paints i.e. a fixed axis joining these two fixed points. Since the moments of the forces of constraints about this axis are all zeros because of the fact that all the forces meet the axis, the moments of the resultant of the system of forces about the axis must be zero for the equilibrium of the body. Hence, when such a body is in equilibrium, the algebraic sum of the moments of the forces about the axis must be zero.

7.6 Astatic Equilibrium

When a system of coplanar forces acts on a particle and keeps it at rest, then the equilibrium will not be disturbed if we turn each force about its point of application through the same angle in the same direction.

But when a number of forces, P_1, P_2,.. P_n act at given points A_1, A_2, ... A_n of a rigid body and keep it in equilibrium, then these forces will not, in general, continue to preserve equilibrium when the body is displaced in any manner, each force still retaining its magnitude, direction and point of application in the body. If for all displacements of the body, the forces continue to preserve equilibrium, then the body is said to be in *astatic equilibrium*.

The simplest example of astatic equilibrium is furnished by a heavy body suspended by a vertical string attachad at its centre of gravity. Here the system of forces is the weights of the particles of the body and the tension of the string. In whatever manner this body may be displaced about its centre of gravity, all these forces will retain their individual magnitudes, directions and points of application and the body will remain at rest.

When a system of forces applied to a body is not in equilibrium, then this system can sometimes be equilibrated astatically by a single applied force of constant magnitude, direction and point of applicotion.

This is always the case for a system of parallel forces. A single force equal and opposite to their resultant applied at the centre of the forces will astatically equilibrate them.

We now prove an important theorem which can be stated as follows:

Theorem: If all the forces in a coplanar system are rotated in the same sense, through the same angle in their plane round their points of application, then their resultant passes through a fixed point in the body. This fixed point is called the *astatic centre.*

Let the coplanar forces $P_1, P_2, \ldots P_n$ act at the points (x_1, y_1), (x_2, y_2) (x_n, y_n) of a body and be inclined to the x-axis at angles $\alpha_1, \alpha_2, \ldots \alpha_n$ respectively. Let X and Y be respectively the algebraic sums of the resolved parts of all the forces along x and y axes respectively.

Then $X = \sum_{r=1}^{n} P_r \cos \alpha_r$ and $Y = \sum_{r=1}^{n} P_r \sin \alpha_r$.

If G be the algebraic sum of the moments of all the forces about the point O, then $G = \sum_{r=1}^{n} (x_r Y_r - y_r X_r) = \sum_{r=1}^{n} P_r (x_r \sin \alpha_r - y_r \cos \alpha_r)$. ...(1)

If (x, y) be the co-ordinates of any point on the line of action of the resultant of the forces, then the equation of the line of action of the resultant is given by

$$y X - x Y + G = 0.$$

Let all the forces be turned through an angle α about their points of application in the same plane and in the same direction so that α_r becomes $\alpha_r + \alpha$ ($r = 1, 2, \ldots n$) and the equation of the line of action of the new resultant becomes

$$yX' - xY' + G' = 0, \qquad \ldots(2)$$

where

$$X' = \sum_{r=1}^{n} P_r \cos(\alpha_r + \alpha) = \sum_{r=1}^{n} P_r \cos \alpha_r \cos \alpha - \sum_{r=1}^{n} P_r \sin \alpha_r \sin \alpha$$
$$= X \cos \alpha - Y \sin \alpha,$$

$$Y' = \sum_{r=1}^{n} P_r \sin(\alpha_r + \alpha) = \sum_{r=1}^{n} P_r \sin \alpha_r \cos \alpha + \sum_{r=1}^{n} P_r \cos \alpha_r \sin \alpha$$
$$= X \sin \alpha + Y \cos \alpha$$

and $G'=\sum_{r=1}^{n} P_r\Big[(x_r \sin(\alpha_r+\alpha)-y_r \cos(\alpha_r+\alpha)\Big]$

$$=\sum_{r=1}^{n} P_r\Big[(x_r(\sin\alpha_r \cos\alpha+\cos\alpha_r \sin\alpha)$$

$$-y_r(\cos\alpha_r \cos\alpha-\sin\alpha_r \sin\alpha)\Big]$$

$$=\sum_{r=1}^{n} \cos\alpha\, P_r(x_r \sin\alpha_r-y_r \cos\alpha_r)$$

$$+\sum_{r=1}^{n} \sin\alpha\, P_r(x_r \cos\alpha_r+y_r \sin\alpha_r)$$

$$=G\cos\alpha+V\sin\alpha,$$

where $V=\sum_{r=1}^{n} P_r(x_r \cos\alpha_r+y_r \sin\alpha_r)$

$$=\sum_{r=1}^{n}(X_r x_r+Y_r y_r),$$

X_r and Y_r being the components of P_r on the x and y-axes respectively.

V is called the *Virial* of the system. Substituting the values of X', Y' and G' in (2), we obtain the equation of the new resultant as
$y(X\cos\alpha-Y\sin\alpha)-x(X\sin\alpha+Y\cos\alpha)+G\cos\alpha+V\sin\alpha=0$

$\Rightarrow \cos\alpha\,(yX-xY+G)-\sin\alpha\,(xX+yY-V)=0.$

This shows that for all values of α, the new resultant passes through the point of intersection of the lines given by

$$xY-yX-G=0$$

$$\text{and } xX+yY-V=0.$$

Solving these equations for x and y, we get

$$\frac{x}{VX+GY}=\frac{y}{-GX+VY}=\frac{1}{X^2+Y^2}$$

$$\Rightarrow x=\frac{VX+GY}{X^2+Y^2} \text{ and } y=\frac{VY-GX}{X^2+Y^2}.$$

Hence, the new resultant always passes through the point whose co-ordinates are

$$\left(\frac{VX+GY}{X^2+Y^2},\quad \frac{VY-GX}{X^2+Y^2}\right)$$

which is a fixed point, since it is independent of α. These are , therefore, the co-ordinates of the astatic centre.

We note that if the forces are in equilibrium originally then $X=0$, $Y=0$ and $G=0$ and after rotation of the forces through an angle α, we have

$X'=0$, $Y'=0$ but $G'=V \sin \alpha$.

Hence, *if the system of coplanar forces which are in equilibrium are rotated through the same angle α in the same plane round their points of application, the new system of forces is equivalent to a couple whose moment is $V \sin \alpha$, where V is a constant independent of α.*

It follows that if the system of forces is in equilibrium both before and after rotatory displacement, then we get four conditions of equilibrium instead of three. These four conditions are $X=0$, $Y=0$, $G=0$ and $V=0$. These are, therefore, the conditions of astatic equilibrium of a rigid body.

7.7 Equilibrium of three forces acting on a rigid body

Theorem : If three coplanar forces acting on a rigid body are in equilibrium, they must either be concurrent or be all parallel to one another.

Let P, Q and R be three coplanar forces which act on a rigid body and are in equilibrium. Since the forces are in equilibrium, one of the forces, say R, must balance the resultant of the other two forces P and Q. Now, since P and Q are coplanar, either P and Q will intersect or they will be parallel to each other. When P and Q intersect at a point, say O, then the algebraic sum of the moments of P, Q and R about O must be zero, since the forces are in equilibrium. But since P and Q meet at O, the moments of P and Q about O will be separately zero and hence the moment of R about O will also be zero. Hence, R must pass through the same point O i.e. P, Q and R must be concurrent.

If P and Q are parrallel to each other, their resultant R must be a parallel force balancing the resultant of P and Q. Hence, R must act in the same line as the line of action of the resultant of P and Q but in the opposite direction. Hence, in this case P, Q and R are all parallel to one another.

Remark 1 : P and Q can never form a couple in this case, for then P, Q and R can not be in equilibrium.

Remark 2: The above theorem gives a *necessary* condition of equilibrium of three coplanar forces, but not *sufficient*. For sufficient condition, when the forces meet at a point, they must also satisfy Lami's theorem or converse of the triangle of forces. When all the three forces are parallel to one another, one being equal and opposite to the resultant of the other two, their algebraic sum must be zero and the moments of any two forces about a point on the line of action of the third force must be equal and opposite.

7.8 Important Trigonometrical Theorems

In many problems of Statics, it is sometimes necessary to find the geometrical configuration of a body in equilibrium under the action of a system of coplanar forces. In such cases, the following two important theorems of Trigonometry are very helpful in deriving the result at once from the geometry of the forces acting on the body.

Theorems: If a straight line *CD* drawn from the vertex *C* of a triangle *ABC* divides the opposite side *AB* into two segments in the ratio m: n, then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

and (ii) $(m+n)\cot\theta = n\cot A - m\cot B$,

where α, β are the angles which *CD* makes with *CA* and *CB* respectively and θ is the angle which *CD* makes with *AB*.

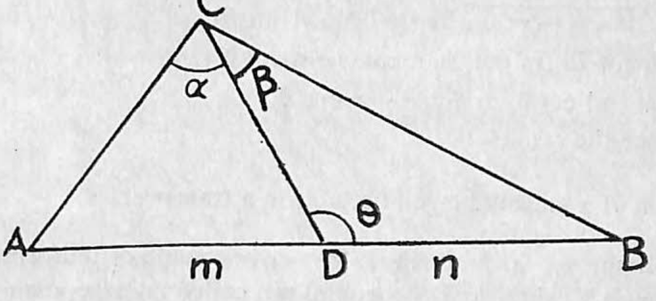

FIG. 90

We have from fig. 90,

$$\frac{m}{n} = \frac{AD}{DB} = \frac{\dfrac{AD}{CD}}{\dfrac{DB}{CD}} = \frac{\dfrac{\sin\alpha}{\sin A}}{\dfrac{\sin\beta}{\sin B}}$$

$$= \frac{\sin\alpha\sin B}{\sin A\sin\beta}. \qquad \ldots(1)$$

Now, to prove (i), we write

$B = 180° - (\beta+\theta)$

and $A = \theta - \alpha$ in (1).

Hence, from (1), we get

$$\frac{m}{n}=\frac{\sin\alpha\sin(\beta+\theta)}{\sin\beta\sin(\theta-\alpha)}=\frac{\sin\alpha(\sin\beta\cos\theta+\cos\beta\sin\theta)}{\sin\beta(\sin\theta\cos\alpha-\cos\theta\sin\alpha)}$$

$=\frac{\cot\theta+\cot\beta}{\cot\alpha-\cot\theta}$ [Dividing both numerator and denominator by $\sin\alpha\sin\beta\sin\theta$]

$\Rightarrow m\cot\alpha-m\cot\theta=n\cot\theta+n\cot\beta$

$\Rightarrow(m+n)\cot\theta=m\cot\alpha-n\cot\beta.$

Now, to prove (ii), we write

$$\alpha=\theta-A$$

and $\beta=180°-(\theta+B)$.

Hence, from (1), we get

$$\frac{m}{n}=\frac{\sin(\theta-A)\sin B}{\sin(\theta+B)\sin A}$$

$$=\frac{(\sin\theta\cos A-\cos\theta\sin A)\sin B}{(\sin\theta\cos B+\cos\theta\sin B)\sin A}$$

$=\frac{\cot A-\cot\theta}{\cot B+\cot\theta}$ [Dividing both numerator and denominator by $\sin\theta\sin A\sin B$]

$\Rightarrow m\cot B+m\cot\theta=n\cot A-n\cot\theta$

$\Rightarrow(m+n)\cot\theta=n\cot A-m\cot B.$

Hence the results.

7.9 Action of a smooth hinge of a joint in a framework

A hinge or a joint in a framework made of uniform bars in the same plane is a cylindrical or a round pin called an axis passing through a small cylindrical hole made at the ends of the bars. If the pin is smooth, the reaction between the two bars or beams connected by it consists of a single force passing through its centre. For, let *PQS* represent a section of the joint connecting two beams [fig. 91]. Then since their surfaces are in contact, either throughout the whole of the circumference or a part of it, there will be normal reactions at the points of contact, *P*, *Q*, *R*,...... Now, since all these forces pass through the centre *O* of the circle, they have a single resultant through this point. Consequently, the reaction in this case consists of a single force through the centre of the joint. By symmetry, the resultant is the same for every section and we get a system of like equal parallel forces acting at every point of the central axis of the cylindrical pin. These like parallel forces give rise to a single resultant acting through the centre of the pin.

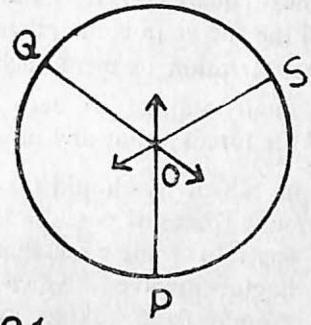

FIG. 91

In a smooth joint connecting two bars, the forces of reaction acting on one bar are equal and opposite to the forces of reaction acting on the other bar for equilibrium. The direction of the reaction on a bar depends on the other forces acting on it.

To find the reaction at a smooth joint in a framework, it is convenient to draw the connected bars separated or disjointed from one another and to draw the forces acting on each bar, the forces of reaction at each joint being resolved in two suitable perpendicular directions. Then we should consider the separate equilibrium of each bar and write down the equations of resolution and the equation of moment for each bar. This method is called *the method of separation of bars.*

In solving problems of framework, we should remember that *if the framework and the forces acting on it are both symmetrical about any line passing through a joint, then the reaction at that joint is perpendicular to the line of symmetry.*

Remark: If a joint be rough, the reactions at the points of contact will not be normal to the cylindrical surface and all the forces at the points of contact will not meet at a point. Therefore, the reactions may reduce to a couple or a single force.

7.10 Working principles of solving problems of equilibrium of a rigid body

In solving problems of equilibrium of a rigid body acted upon by a system of coplanar forces, we should first of all obtain three equations connecting the unknown forces and the angles as follows:

(i) We should first of all equate to zero the algebraic sum of the resolved parts of all the forces in some convenient direction.

(ii) We should next equate to zero the algebraic sum of the resolved parts of all the forces in some other direction usually perpendicular to the direction as mentioned in (i)

(iii) We should finally equate to zero the algebraic sum of the moments of the forces about any point in the plane of the forces.

The point about which we should take moments is usually chosen so as to exclude as many forces as possible i.e. the point through which most ot the forces pass. In some cases there exist geometrical relations between lengths or angles involved, which give additional equations to those obtained by resolving and taking moments. In many problems, the difficulties are not really in the mechanical principles of the problem, but in the geometrical and trigonometrical knowledge required to obtain the result asked for.

When a body is in equilibrium under the action of three forces only, we should first of all see whether two of the forces meet or not. If two forces are parallel, then the third force must be parallel to them. On the other hand, if two forces meet at a point, the third force will also pass through the same point. In the latter case, Lami's theorem or converse of Triangle of Forces is often helpful. In many cases, equating to zero the algebraic sum of the moments of the forces about a convenient point is also helpful. Drawing of an incorrect figure according to the specifications of a problem sometimes leads to an absurd conclusion. In addition to the above, the following points should also be remembered:

(a) The tension of the same string is the same throughout its length. If there is a knot in a string, the tensions of the string on the two sides of the knot will be, in general, different.

(b) When two smooth bodies are in contact, the reaction of one on the other will be along the common normal to the two bodies.

(c) When two bodies (usually rods) are connected by a smooth joint or hinge at a point and are freely movable about it, reactions on them will be equal and opposite, but the direction of the reaction depends on the other forces acting on the body concerned.

(d) If two bodies are rigidly joined together by a hinge and if we consider the equilibrium of the whole system, then the internal reactions at the joint should not be considered, since they balance one another. But if we consider the equilibrium of each body separately, then the internal reactions must be considered. All these points will be clear from the solved examples given below.

7.11 Solved Examples

(a) Reduction of coplanar forces acting on a rigid body

Ex. 1. Forces P, Q and R act along the sides BC, CA and AB of the triangle ABC in the directions indicated by the order of the letters. If F be the magnitude of their resultant, show that
$F^2=P^2+Q^2-2QR\cos A-2RP\cos B-2PQ\cos C$.

Let θ be the angle made by the resultant F with BC. Then resolving along BC and perpendicular to it, we have

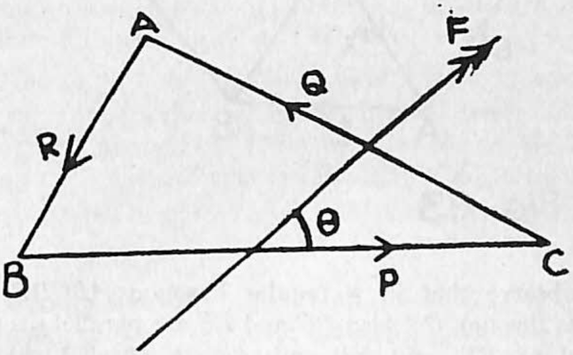

FIG. 92

$$F\cos\theta=P-Q\cos C-R\cos B \qquad \ldots(1)$$

$$\text{and } F\sin\theta=Q\sin C-R\sin B \qquad \ldots(2)$$

Hence, squaring (1) and (2) and adding, we get

$$F^2=P^2+Q^2\cos^2 C+R^2\cos^2 B-2PQ\cos C-2PR\cos B$$
$$+2QR\cos C\cos B+Q^2\sin^2 C+R^2\sin^2 B-2QR\sin B\sin C$$
$$=P^2+Q^2+R^2-2PQ\cos C-2PR\cos B+2QR\cos(B+C)$$
$$=P^2+Q^2+R^2-2QR\cos A-2RP\cos B-2PQ\cos C.$$

Ex. 2. $ABCDEF$ is a regular hexagon of which O is the centre. Forces of magnitude 1, 2, 3, 4, 5 and 6 act along AB, CB, CD, ED, EF and AF respectively in the directions indicated by the order of the letters.

Reduce the system to a force at O and a couple and find the point in AB through which the single resultant passes.

[I.C.S. 1938]

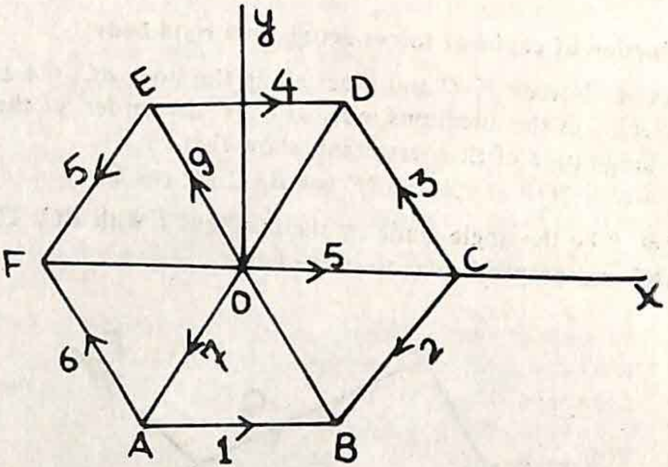

FIG. 93

We observe that in a regular hexagon $ABCDEF$, AD, BE and CF will pass through O. Also BC and FE are parallel to AD, CD and AF are parallel to BE, and ED and AB are parallel to FC. We can now replace each force by a parallel force at O together with a couple. Thus the forces 3 and 6 along CD and AF respectively are equivalent to a force $(6+3)=9$ along OE together with a couple of moment $3P-6p=-3p$, where p is the perpendicular distance of any side from O. Similarly, the forces 1 along AB and 4 along ED are equivalent to a force $(4+1)=5$ along OC together with a couple of moment $p-4p=-3p$, and forces 2 along CB and 5 along EF are equivalent to a single force $(2+5)=7$ along OA together with a couple of moment $5p-2p=3p$. The algebraic sum of the moments of all these couples $=G=-3p-3p+3p=-3p$.

We have now three forces 5 acting along OC, 9 along OE and 7 along OA.

We now take OC as the x-axis and a line perpendicular to OC as the y-axis. Resolving these three forces along x and y axes, we get

X=algebraic sum of the resolved parts of the forces along OX

$$=5-9\cos 60^\circ-7\cos 60^\circ=5-\frac{9}{2}-\frac{7}{2}=-3$$

and Y=algebraic sum of the resolved parts of the forces along Oy

$$=9 \sin 60° - 7 \sin 60° = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}.$$

Hence, the resultant $R=\sqrt{X^2+Y^2}=\sqrt{12}=2\sqrt{3}$.

If (x, y) be any point on the line of action of the resultant, then its equation is

$$xY - yX - G = 0$$

$$\Rightarrow \sqrt{3}\, x + 3y + 3p = 0. \qquad \text{...(1)}$$

Also the equation of the line AB is $y=-p$. Putting $y=-p$ in (1), we get $x=0$.

But $(0, -p)$ are the co-ordinates of the middle point of AB. Hence, the single resultant obtained by the combination of the resultant and the couple passes through the middle point of AB.

Ex. 3. A system of forces in the plane of a triangle ABC is equivalent to a single force at A, acting along the internal bisector of the triangle ABC, and a couple of moment G_1. If the moment of the system about B and C are G_2 and G_3, respectively, prove that $(b+c)\,G_1 = bG_2 + cG_3$, where a, b and c are the lengths of the sides BC, CA and AB respectively of the triangle ABC.

Let us choose the point A as the origin. AB as the x-axis and a line perpendicular to it as the y-axis. Let the single force be R, and X and Y its components along x and y-axes respectively.

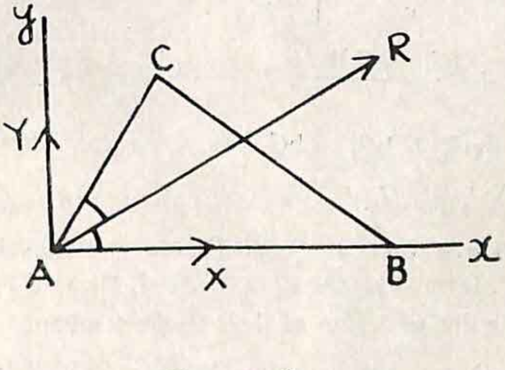

FIG. 94

Taking moments about B and C, we get

$$G_2 = G_1 - cY \qquad \text{...(1)}$$

and $G_3 = G_1 + b \sin A.\, X - b \cos A.\, Y$ ...(2)

Also $X = R \cos \dfrac{A}{2}$ ...(3)

and $Y = R \sin \dfrac{A}{2}$. (4)

From (3) and (4), we have $\tan \dfrac{A}{2} = \dfrac{Y}{X} \Rightarrow X = Y \cot \dfrac{A}{2}$. ...(5)

From (1), $Y = \dfrac{G_1 - G_2}{c}$ and hence from (5),

$$X = \frac{G_1 - G_2}{c} \cot \frac{A}{2}.$$

Substituting the values of X and Y in (2), we get

$$G_3 = G_1 + \frac{(G_1 - G_2)b}{c}\left[\sin A \cot \frac{A}{2} - \cos A\right]$$

$$= G_1 + \frac{(G_1 - G_2)b}{c}\left[\frac{2 \sin \frac{A}{2} \cos^2 \frac{A}{2}}{\sin \frac{A}{2}} - \cos A\right]$$

$$= G_1 + \frac{(G_1 - G_2)b}{c}\left[1 + \cos A - \cos A\right]$$

$$= G_1 + \frac{(G_1 - G_2)b}{c}$$

$$\Rightarrow cG_3 = cG_1 + bG_1 - bG_2$$

$$\Rightarrow (b+c)\, G_1 = bG_2 + cG_3.$$

Ex. 4. Three forces P, Q and R act along the sides of the triangle, taken in order, formed by the lines $x+y=1$, $y-x=1$ and $y=2$. Find the equation of the line of action of their single resultant. [Nagpur '1954]

Let the three given equations be those of the sides AB, BC and CA of the triangle ABC. Let the forces P, Q and R act along the sides AB, BC and CA respectively of the triangle.

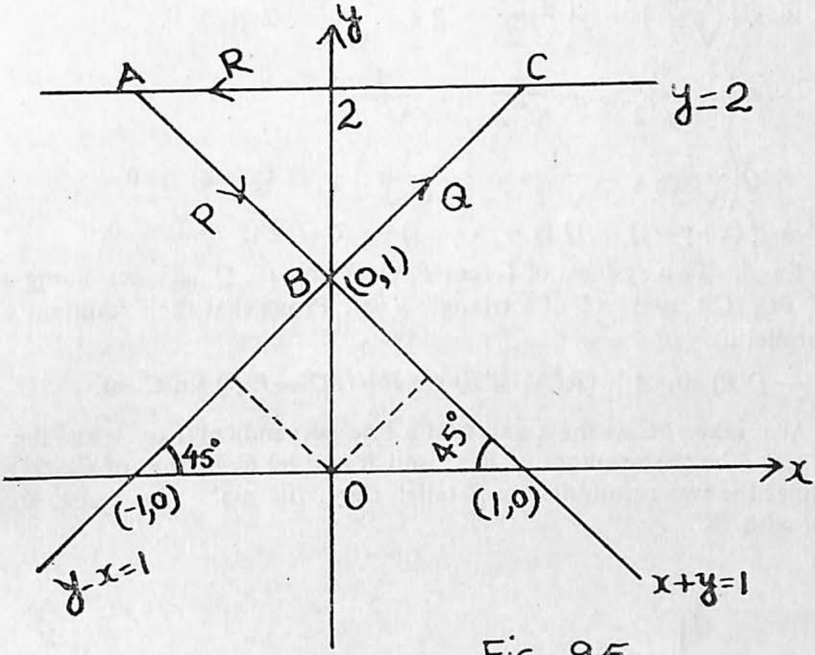

FIG. 95

Resolving all the forces along x and y-axes, we get

X = algebraic sum of the resolved parts along Ox

$$= P \cos 45^\circ + Q \cos 45^\circ - R$$

$$= \frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}} - R.$$

Y = algebraic sum of the resolved parts along Oy

$$= - P \sin 45^\circ + Q \sin 45^\circ = - \frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}}.$$

Taking moments of all the forces about O, we get

G = algebraic sum of the moments of the forces about O

$$= - P.\ 1 \sin 45^\circ - Q.\ 1 \sin 45^\circ + 2R$$

$$= - \frac{P}{\sqrt{2}} - \frac{Q}{\sqrt{2}} + 2R.$$

Hence, the equation of the line of action of the resultant force is $xY - yX - G = 0$

$$\Rightarrow x\left(\frac{Q-P}{\sqrt{2}}\right)-y\left(\frac{Q+P}{\sqrt{2}}-R\right)+\frac{P+Q}{\sqrt{2}}-2R=0.$$

$$\Rightarrow P\left(-\frac{1}{\sqrt{2}}y-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}\right)$$

$$+Q\left(\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}\right)+R(y-2)=0$$

$$\Rightarrow P(x+y-1)+Q(y-x-1)-R\sqrt{2}(y-2)=0.$$

Ex. 5. Two systems of forces P, Q, R and P', Q', R' act along the sides BC, CA and AB of a triangle ABC. Prove that their resultant will be parallel if

$$(QR'-Q'R)\sin A+(RP'-R'P)\sin B+(PQ'-P'Q)\sin C=0.$$

We take BC as the x-axis and a line perpendicular to it as the y-axis. Let F be the resultant of P, Q and R and let F' be that of P', Q' and R'. Since the two resultants are parallel, they will make the same angle, say θ, with BC.

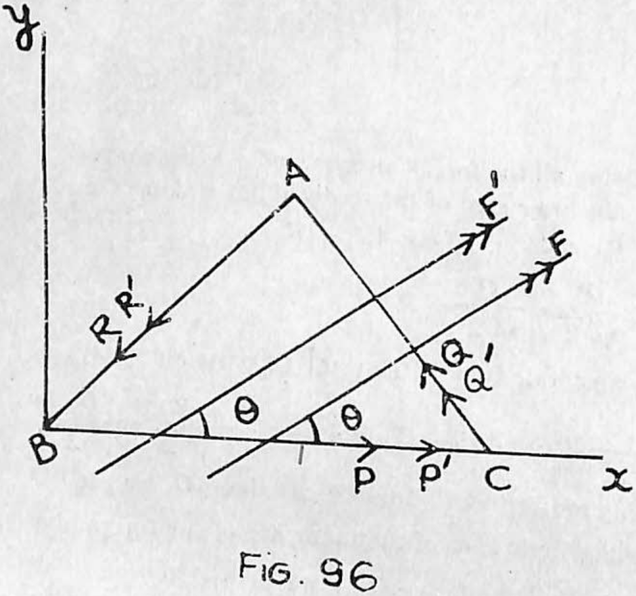

FIG. 96

Resolving the first system of forces along x and y-axes, we get

$F\cos\theta = P - Q\cos C - R\cos B$ along Bx

and $F\sin\theta = Q\sin C - R\sin B$ along By.

Hence,

$$\tan\theta = \frac{Q\sin C - R\sin B}{P - Q\cos C - R\cos B}. \qquad \ldots(1)$$

Similarly, for the second system of forces, we get

$$\tan\theta = \frac{Q'\sin C - R'\sin B}{P' - Q'\cos C - R'\cos B}. \qquad \ldots(2)$$

From (1) and (2), we get

$$(P - Q\cos C - R\cos B)(Q'\sin C - R'\sin B)$$

$$= (Q\sin C - R\sin B)(P' - Q'\cos C - R'\cos B)$$

$$\Rightarrow PQ'\sin C - PR'\sin B - QQ'\cos C\sin C$$

$$+ QR'\cos C\sin B - RQ'\cos B\sin C + RR'\cos B\sin B$$

$$= QP'\sin C - QQ'\sin C\cos C - QR'\sin C\cos B$$

$$- RP'\sin B + RQ'\sin B\cos C + RR'\sin B\cos B$$

$$\Rightarrow (PQ' - QP')\sin C + (RP' - R'P)\sin B$$

$$+ QR'(\cos C\sin B + \sin C\cos B)$$

$$- RQ'(\cos B\sin C + \sin B\cos C) = 0$$

$$\Rightarrow (PQ' - QP')\sin C + (RP' - R'P)\sin B$$

$$+ (QR' - RQ')\sin(B + C) = 0$$

$$\Rightarrow (QR' - Q'R)\sin A + (RP' - R'P)\sin B + (PQ' - P'Q) \times \sin C = 0.$$

$$[\because B + C = \pi - A]$$

Ex. 6. The sum of the moments of a system of coplanar forces about each of three non-collinear points in the same plane as that of the forces is equal to the sum of the moments of another system about the same points. Prove that the two systems are equivalent.

Hence show that any system of coplanar forces may be replaced by three forces acting round the three sides of a given triangle ABC in the same plane as that of the forces. [Allahabad 1957]

First Part :

Let X and Y be the algebraic sums of the resolved parts of one system of forces along two mutually perpendicular directions Ox and Oy respectively and let G be the algebraic sum of the moments of the forces about O. Let X_1, Y_1 and G_1 be the corresponding quantities for the second system of forces.

Let the co-ordinates of three non-collinear points be (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Then the algebraic sum of the moments of the forces in the first system about (x_1, y_1) is $G - x_1 Y + y_1 X$ and that for the second system is $G_1 - x_1 Y_1 + y_1 X_1$.

According to the given condition, we have

$$G - x_1 Y + y_1 X = G_1 - x_1 Y_1 + y_1 X_1$$

$$\Rightarrow G - G_1 - x_1 (Y - Y_1) + y_1 (X - X_1) = 0. \tag{1}$$

Similarly, taking moments about (x_2, y_2) and (x_3, y_3), we get

$$G - G_1 - x_2 (Y - Y_1) + y_2 (X - X_1) = 0 \qquad \ldots(2)$$

and $G - G_1 - x_3 (Y - Y_1) + y_3 (X - X_1) = 0$. $\qquad \ldots(3)$

Solving for $G - G_1$, $Y - Y_1$ and $X - X_1$ from (2) and (3), we get

$$\frac{G - G_1}{x_3 y_2 - x_2 y_3} = \frac{Y - Y_1}{y_2 - y_3} = \frac{X - X_1}{x_2 - x_3} = K, \text{ say.}$$

$$\left.\begin{array}{l} \therefore G - G_1 = K (x_3 y_2 - x_2 y_3),\ Y - Y_1 = K (y_2 - y_3) \\ \text{and } X - X_1 = K (x_2 - x_3). \end{array}\right\} \ldots (A)$$

Substituting these values of $G - G_1$, $Y - Y_1$ and $X - X_1$ in (1), we get

$$K [x_1 (y_2 - y_3) - y_1 (x_2 - x_3) + (x_2 y_3 - x_3 y_2)] = 0$$

$$\Rightarrow K \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

But since the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are non-collinear, hence

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \neq 0.$$

Hence, K must be zero.

Hence, from (A), we get $G = G_1$, $X = X_1$ and $Y = Y_1$.

In other words, the two systems of forces are equivalent.

Second part :

Let ABC be a triangle and let the algebraic sums of the moments of a system of coplanar forces about A, B and C be G_1, G_2 and G_3 respectively. We consider another system of three forces P, Q and R acting along the sides BC, CA and AB respectively of a triangle ABC and let

G_1', G_2' and G_3' be the algebraic sums of the moments of P, Q and R about A, B and C respectively. If these two systems of forces are equivalent, then as in the first part,

$$G_1 = G_1',\ G_2 = G_2' \text{ and } G_3 = G_3'.$$

But $G_1' = p_1P$, $G_2' = p_2Q$ and $G_3' = p_3R$,

where p_1, p_2 and p_3 are respectively the lengths of the perpendiculars drawn from A, B and C to the opposite sides.

Hence, $p_1P = G_1$, $p_2Q = G_2$ and $p_3R = G_3$

$$\Rightarrow P = \frac{G_1}{p_1},\ Q = \frac{G_2}{p_2} \text{ and } R = \frac{G_3}{p_3}.$$

Thus G_1, G_2 and G_3 of one system of forces being known and p_1, p_2 and p_3 being known, we can determine another system of forces P, Q and R acting along the sides of a triangle.

Ex. 7. Moments of the resultant R of a system of coplanar forces about three points O, A and B lying in the plane of the forces are G, $G+J_1$ and $G+J_2$ respectively. If, referred to O as origin, the polar co-ordinates of A and B be (r_1, θ_1) and (r_2, θ_2) respectively, show that

$$R^2 \sin^2(\theta_1-\theta_2) = \frac{J_1^2}{r_1^2} + \frac{J_2^2}{r_2^2} - 2\frac{J_1J_2\cos(\theta_1-\theta_2)}{r_1r_2}. \quad \text{[C.H. 1954]}$$

We take O as the origin and any two perpendicular lines Ox and Oy as the co-ordinate axes. Since the polar co-ordinates of A and B are given to be (r_1, θ_1) and (r_2, θ_2) respectively, their rectangular co-ordinates

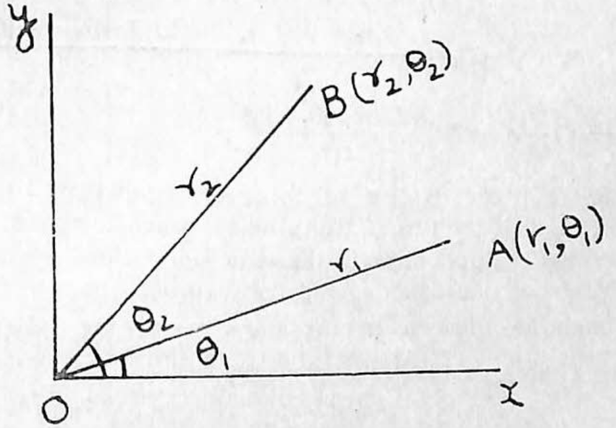

FIG. 97

are A : $(r_1 \cos\theta_1, r_1 \sin\theta_1)$ and B : $(r_2 \cos\theta_2, r_2 \sin\theta_2)$. Since the algebraic sum of the moments of the system of forces about A is $G + J_1$, hence

$$G - Xr_1 \sin\theta_1 + Yr_1 \cos\theta_1 = G + J_1$$

$$\Rightarrow Xr_1 \sin\theta_1 - Yr_1 \cos\theta_1 + J_1 = 0, \qquad \ldots(1)$$

where X and Y are the algebraic sums of the resolved parts of the forces along x and y axes respectively.

Similarly, taking moments about B, we have

$$G - Xr_2 \sin\theta_2 + Yr_2 \cos\theta_2 = G + J_2$$

$$\Rightarrow Xr_2 \sin\theta_2 - Yr_2 \cos\theta_2 + J_2 = 0\,. \qquad \ldots(2)$$

Solving for X and Y from (1) and (2), we get

$$\frac{X}{J_1r_2 \cos\theta_2 - J_2r_1 \cos\theta_1} = \frac{Y}{J_1r_2 \sin\theta_2 - J_2r_1 \sin\theta_1}$$

$$= \frac{1}{r_1r_2 (\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2)} = \frac{1}{-r_1r_2 \sin(\theta_1 - \theta_2)}$$

Hence, $X = \dfrac{J_1r_2 \cos\theta_2 - J_2r_1 \cos\theta_1}{-r_1r_2 \sin(\theta_1 - \theta_2)}$

and $Y = \dfrac{J_1r_2 \sin\theta_2 - J_2r_1 \sin\theta_1}{-r_1r_2 \sin(\theta_1 - \theta_2)}$.

Hence, $R^2 = X^2 + Y^2$

$$= \frac{(J_1r_2 \cos\theta_2 - J_2r_1 \cos\theta_1)^2 + (J_1r_2 \sin\theta_2 - J_2r_1 \sin\theta_1)^2}{r_1^2r_2^2 \sin^2(\theta_2 - \theta_1)}$$

$$\Rightarrow R^2 \sin^2(\theta_1 - \theta_2) = \frac{J_1^2r_2^2 + J_2^2r_1^2 - 2J_1J_2\, r_2r_1 \cos(\theta_1 - \theta_2)}{r_1^2r_2^2}$$

$$= \frac{J_1^2}{r_1^2} + \frac{J_2^2}{r_2^2} - 2.\frac{J_1J_2 \cos(\theta_1 - \theta_2)}{r_1r_2}.$$

Ex. 8. *A* plane system of forces is equivalent to a couple of moment M, and if the forces are turned through a right angle about the respective points of application in the same sense, they are equivalent to a couple N. Prove that when each force is turned about its point of application through an angle α in the same sense, the system will be in equilibrium if $\tan\alpha = -\dfrac{M}{N}$.

[*C.H.* 1959]

Show that if the forces be turned through an angle $2\tan^{-1}\dfrac{N}{M}$, the system is equivalent to a couple of moment M.

Let $P_1, P_2 \ldots\ldots P_n$ be the system of forces acting at the points (x_1, y_1) $(x_2, y_2), \ldots\ldots (x_n, y_n)$ and making angles $\theta_1, \theta_2 \ldots \theta_n$ with the x-axis. Let X_r and Y_r be the components of P_r $(r=1,2\ldots n)$ along x and y-axes respectively and X and Y be the algebraic sums of the resolved parts of the forces along x and y axes respectively. Let G be the algebraic sum of the moments of the forces about the origin O.

Let X, Y, G become X', Y', G' and X'', Y'', G'', when all the forces are turned through a right angle and any angle α respectively.

Then we have

$X_r = P_r \cos\theta_r$, $Y_r = P_r \sin\theta_r$ so that

$$X = \sum_{r=1}^{n} X_r = \sum_{r=1}^{n} P_r \cos\theta_r,\ Y = \sum_{r=1}^{n} Y_r = \sum_{r=1}^{n} P_r \sin\theta_r$$

$$\text{and } G = \sum_{r=1}^{n} (x_r Y_r - y_r X_r) = \sum_{r=1}^{n} (x_r \sin\theta_r - y_r \cos\theta_r) P_r.$$

Since the system of forces reduces to a couple of moment M only, hence we have

$$X = Y = 0 \Rightarrow \sum_{r=1}^{n} P_r \cos\theta_r = \sum_{r=1}^{n} P_r \sin\theta_r = 0 \qquad \ldots(1)$$

$$\text{and } G = M \Rightarrow \sum_{r=1}^{n} (x_r \sin\theta_r - y_r \cos\theta_r) P_r = M \qquad \ldots(2)$$

$$\text{Now, } X' = \sum_{r=1}^{n} P_r \cos\left(\theta_r + \frac{\pi}{2}\right) = -\sum_{r=1}^{n} P_r \sin\theta_r = 0 \left[\text{by (1)}\right]$$

$$Y' = \sum_{r=1}^{n} P_r \sin\left(\theta_r + \frac{\pi}{2}\right) = \sum_{r=1}^{n} P_r \cos\theta_r = 0 \left[\text{by (1)}\right]$$

$$\text{and } G' = \sum_{r=1}^{n} \left[x_r \sin\left(\theta_r + \frac{\pi}{2}\right) - y_r \cos\left(\theta_r + \frac{\pi}{2}\right)\right] P_r$$

$$= \sum_{r=1}^{n} (x_r \cos\theta_r + y_r \sin\theta_r) P_r.$$

Hence, when all the forces are turned through a right angle, then the system of forces, reduces to a couple of moment $G' = N$ so that

$$\sum_{r=1}^{n} (x_r \cos \theta_r + y_r \sin \theta_r) P_r = N. \qquad \ldots(3)$$

When the forces are turned through an angle α, we have

$$X'' = \sum_{r=1}^{n} P_r \cos (\theta_r + \alpha)$$

$$= \cos \alpha \sum_{r=1}^{n} P_r \cos \theta_r - \sin \alpha \sum_{r=1}^{n} P_r \sin \theta_r = 0 \text{ [by (1)]}$$

$$Y'' = \sum_{r=1}^{n} P_r \sin (\theta_r + \alpha)$$

$$= \sin \alpha \sum_{r=1}^{n} P_r \cos \theta_r + \cos \alpha \sum_{r=1}^{n} P_r \sin \theta_r = 0 \qquad \text{[by (1)]}$$

and

$$G'' = \sum_{r=1}^{n} P_r [x_r \sin (\theta_r + \alpha) - y_r \cos (\theta_r + \alpha)]$$

$$= \sum_{r=1}^{n} P_r [x_r (\sin \theta_r \cos \alpha + \cos \theta_r \sin \alpha)$$

$$- y_r (\cos \theta_r \cos \alpha - \sin \theta_r \sin \alpha)]$$

$$= \cos \alpha \sum_{r=1}^{n} P_r (x_r \sin \theta_r - y_r \cos \theta_r)$$

$$+ \sin \alpha \sum_{r=1}^{n} P_r (x_r \cos \theta_r + y_r \sin \theta_r)$$

$$= M \cos \alpha + N \sin \alpha. \qquad \text{[by (2) and (3)]}$$

Now if the system of forces be in equilibrium when turned through an angle α, we must have $G'' = 0$

$$\Rightarrow M \cos \alpha + N \sin \alpha = 0 \Rightarrow \tan \alpha = -\frac{M.}{N}$$

If $\alpha = 2 \tan^{-1} \dfrac{N}{M} \Rightarrow \dfrac{N}{M} = \tan \dfrac{\alpha,}{2}$ then

$$G'' = M \cos \alpha + N \sin \alpha = M. \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} + N. \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$\Rightarrow G''=M.\frac{1-\frac{N^2}{M^2}}{1+\frac{N^2}{M^2}}+N.\frac{2\frac{N}{M^2}}{1+\frac{N^2}{M^2}}$$

$$=\frac{M(M^2-N^2)+2N^2M}{M^2+N^2}$$

$$=M.\frac{M^2+N^2}{M^2+N^2}=M.$$

Hence, in this case the system reduces to a couple of moment M.

Ex. 9. A system of coplanar forces has the total moments H, $2H$ respectively about the points whose co-ordinates are (2a, 0) and (0, a) referred to fixed rectangular axes. The total resolved parts of the forces along the line y=x vanishes. Find the points in which the line of action of the resultant meets the co-ordinate axes. [C.H. 1961]

Let the resultant of the system of forces be R whose line of action meets the x-axis at (b, 0).

Let $-X$ and Y be its components along x and y-axes respectively. Taking moments of the forces about the point (0, a), we get

$$Y\text{b}-X\text{a}=2H. \tag{1}$$

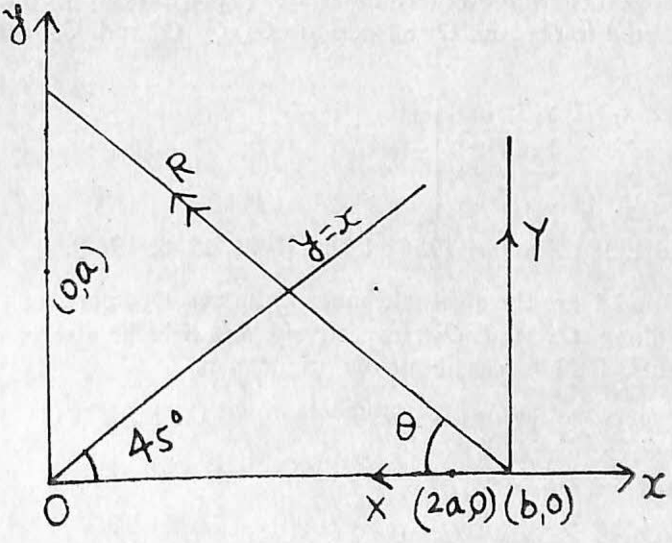

Fig. 98

Taking moments of the forces about the point $(2a, 0)$, we have, $-Y(2a-b)=H$. ...(2)

Resolving the components X and Y along the line $y=x$, we get

$$Y\cos 45^\circ - X\cos 45^\circ = 0 \Rightarrow Y = X. \quad \text{...(3)}$$

From (1) and (3), we have

$$Y(b-a)=2H. \quad \text{...(4)}$$

Dividing (2) by (4), we get

$$\frac{b-2a}{b-a}=\frac{1}{2}\Rightarrow 2b-4a-b+a=0 \Rightarrow b=3a.$$

Hence, the line of action of the resultant intersects the x-axis at the point $(3a, 0)$.

Since $Y=X$, hence the angle θ made by the line of action of the resultant with negative x-axis is given by $\tan\theta=\frac{Y}{X}=1\Rightarrow\theta=45^\circ$.

Hence, the intercepts made by the line of action of the resultant on the x and y-axes will be equal. Hence, the line of action of the resultant R will cut the y-axis at the point $(0, 3a)$.

Ex. 10. The algebraic sum of the moments of a system of forces about the points whose co-ordinates are (x_1, y_1), (x_2, y_2), (x_3, x_3) and (x_4, y_4) referred to Ox and Oy as axes are G_1, G_2, G_3 and G_4 respectively. Show that

$$\begin{vmatrix} 1 & x_1 & y_1 & G_1 \\ 1 & x_2 & y_2 & G_2 \\ 1 & x_3 & y_3 & G_3 \\ 1 & x_4 & y_4 & G_4 \end{vmatrix}=0.$$

[Allahabad 1938 ; Bombay 1954 ; Delhi Hons. 1958, 1962].

If X and Y are the algebraic sums of the resolved parts of the system of forces along Ox and Oy respectively, and G is the algebraic sum of the moments of the forces about the origin, then

$$\begin{aligned} G_1 &= \text{algebraic sum of the moments about } (x_1, y_1) \\ &= G - x_1 Y + y_1 X \\ &\Rightarrow G - x_1 Y + y_1 X - G_1 = 0 \end{aligned}$$

Similarly, $G-x_2 Y+y_2 X-G_2=0$

$$G-x_3 Y+y_3 X-G_3=0$$

and $G-x_4 Y+y_4 X-G_4=0$.

Eliminating G, X, Y from these four equations, we get

$$\begin{vmatrix} 1 & x_1 & y_1 & G_1 \\ 1 & x_2 & y_2 & G_2 \\ 1 & x_3 & y_3 & G_3 \\ 1 & x_4 & y_4 & G_4 \end{vmatrix} = 0.$$

Ex. 11. The algebraic sums of the moments of a system of forces about three vertices A, B, C of an equilateral triangle whose sides are 2 units long are 10, 20 and 10 units; find the magnitude of the resultant force and the points where its line of action intersects AB and AC,

Let X and Y be the algebraic sums of the resolved parts of the system of forces along Ax and Ay respectively, A being chosen as the origin and axes Ax and Ay are perpendicular to each other [fig. 99]. Let G be the algebraic sum of the moments of the forces about A. Then $G=10$.

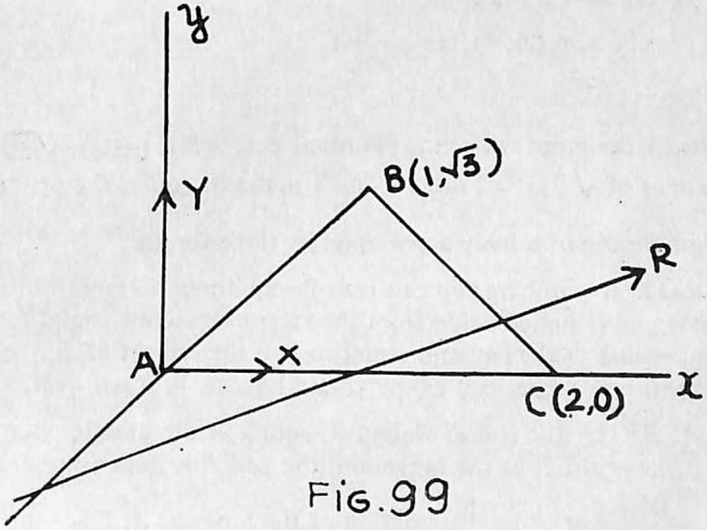

FIG. 99

The co-ordinates of B are (2 cos 60°, 2 sin 60°)

$=(1, \sqrt{3})$ and those of C are (2, 0).

Taking moments about B, we have

$$20=G-Y+\sqrt{3}\,X=10-Y+\sqrt{3}X$$

$$\Rightarrow \sqrt{3}X-Y-10=0. \qquad \ldots(1)$$

Taking moments of the forces about C, we have

$$-10=10-2Y \Rightarrow Y=10.$$

Hence, from (1), $X=\frac{20}{\sqrt{3}}$.

If R be the resultant force, then

$$R=\sqrt{X^2+Y^2}=\sqrt{100+\frac{400}{3}}=10\sqrt{\frac{7}{3}}.$$

The equation of the line of action of the resultant is

$$xY-yX-G=0$$

$$\Rightarrow 10x-\frac{20}{\sqrt{3}}y-10=0 \Rightarrow x-\frac{2}{\sqrt{3}}y-1=0 \quad \ldots(2)$$

Putting $y=0$, we get $x=1$. Hence, the resultant passes through the middle point of AC. Again, the equation of the line AB is

$$y=\tan 60^\circ.\, x=\sqrt{3}x.$$

Putting $y=\sqrt{3}\,x$ in (2), we get $x=-1$.

Hence, $y=-\sqrt{3}$.

Hence, the point where the resultant cuts AB is $(-1,-\sqrt{3})$ which is at a distance of $\sqrt{3+1}=2$ units from A in the direction BA produced.

(b) Equilibrium of a body acted upon by three forces :

Ex. 12. A uniform rod can turn freely about a smooth hinge at one of its ends and is pulled aside from the vertical by a horizontal force acting at the other end of the rod and equal to half the weight of the rod. Find the inclination of the rod to the vertical at which it will rest.

Let AB be the rod of weight W acting at the middle point G of the rod. A is the position of the hinge and the end B is pulled horizontally by a force $\frac{W}{2}$. Let R be the reaction of the hinge at A. Since the rod is in equilibrium under the action of these three forces only, the reaction of the hinge must pass through C, the point of intersection of the lines of action of the horizontal force $\frac{W}{2}$ and the vertical force W. Let the vertical through A intersect BC produced at O so that $\angle AOB=90^\circ$.

Let $\angle ABO=\theta$ and $\angle ACO=\phi$.

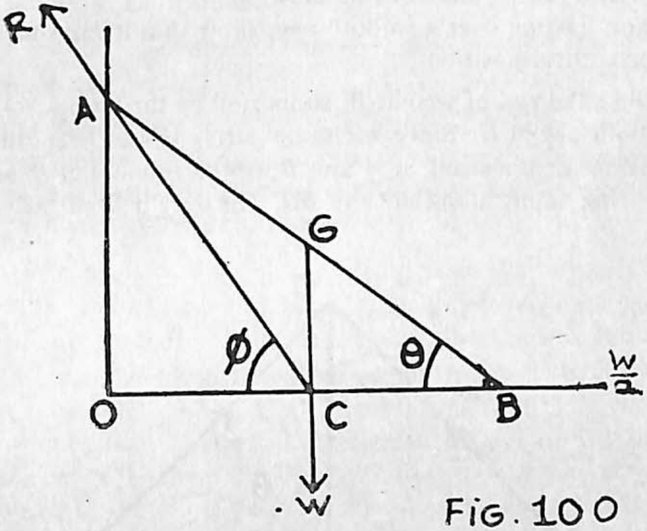

FIG. 100

Since the rod is in equilibrium under three forces meeting at C, we have, by Lami's theorem,

$$\frac{W}{\sin \angle ACB}=\frac{\frac{W}{2}}{\sin \angle ACD}\Rightarrow \frac{1}{\sin\,(180^\circ-\phi)}=\frac{1}{2\sin\,(90^\circ+\phi)}$$

$$\Rightarrow \sin\phi=2\cos\phi.$$

Since ϕ cannot be equal to $\frac{\pi}{2}$ so long as the horizontal force exists, we have

$$\tan\phi=2$$

$$\Rightarrow \frac{AO}{OC}=2\Rightarrow AO=2OC. \qquad \ldots(1)$$

Now since G is the middle point of AB and $GC \parallel AO$, hence $OC=CB$ so that $OB=2OC$.

Hence, from (1)

$AO=OB$ and $\angle ABO=\theta=45^\circ$.

Hence, the inclination of the rod to the vertical is 45°.

Alternative method :

Since the three forces are parallel to the sides of the triangle AOC, we have, by the converse of the triangle of forces,

$$\frac{W}{AO}=\frac{\frac{W}{2}}{OC}\Rightarrow AO=2OC=OB$$

$$\therefore\ \angle ABO=45^\circ.$$

Ex. 13. If a unifrom heavy rod be supported by a string fastened at its ends and passing over a smooth peg, show that it can only rest in a horizontal or vertical position.

Let AB be the rod of weight W supported by the string AOB passing over the smooth peg at O. Since the same string passes over the smooth peg, the tensions of the string at A and B will be equal. Let T be the tension of the string acting along AO and BO. The weight W of the rod passes through G, the middle point of the rod AB and it acts vertically downwards. Since the rod is in equilibrium under these three forces only, the vertical through G must pass through O i.e. GO must be vertical.

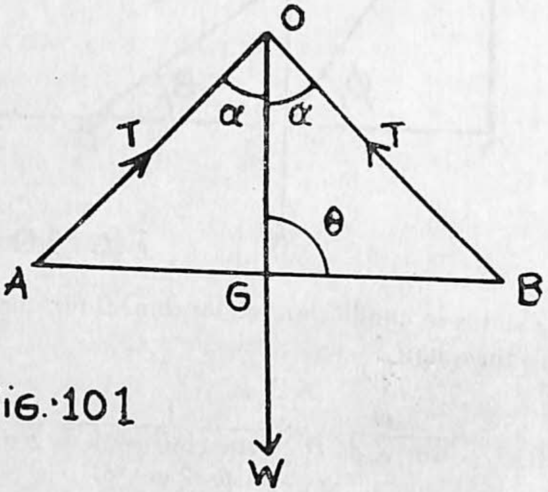

Fig. 101

Now the resultant of two equal forces T, T must bisect the angle between them. Since this resultant force balances W, the resultant of T, T must be along GO i.e. $\angle AOG=\angle GOB$. Now, let $\angle OGB=\theta$, $\angle AOG=\angle GOB=\alpha$.

We have

$$\frac{AG}{\sin\alpha}=\frac{OG}{\sin A}\Rightarrow\frac{AG}{OG}=\frac{\sin\alpha}{\sin(\theta-\alpha)}\text{ and}$$

$$\frac{BG}{\sin\alpha}=\frac{OG}{\sin B}\Rightarrow\frac{BG}{OG}=\frac{\sin\alpha}{\sin[180^\circ-(\theta+\alpha)]}\quad\left[\because AG=BG\right]$$

$$\Rightarrow\sin\alpha\,[\,\sin\{180^\circ-(\theta+\alpha)\}-\sin(\theta-\alpha)]=0.$$

$\therefore$ Either $\sin\alpha=0$ whence $\alpha=0$

or, $\sin[180^\circ-(\theta+\alpha)]=\sin(\theta-\alpha)$

$\Rightarrow 180° - (\theta + \alpha) = \theta - \alpha \Rightarrow 2\theta = 180° \Rightarrow \theta = 90°$.

When $\theta = 90°$, the rod AB is horizontal and when $\alpha = 0$, the rod is vertical. Hence the result.

Ex. 14. A uniform rod of weight W and length $2l$ has one end against a smooth vertical wall and rests at an inclination of 45° with the vertical upon a smooth rail parallel to the wall. Find the distance of the rail from the wall and the reactions.

Let AB be the rod of weight W acting vertically downwards through G, its middle point. The end A is simply pressed against a smooth vertical wall so that the reaction R of the wall on the rod will be perpendicular to the wall and hence the reaction R must be horizontal. Let these two horizontal and vertical forces meet at O. The position of the rail will be such that the normal reaction of the rail on the rod may pass through O. Let C be the point of contact of the rail with the rod so that the reaction R_1 of the rail is perpendicular to AB and it passes through O.

[Note that the equilibrium of the rod is not possible if the point of contact of the rail with the rod lies beyond AG or if the rod AB remains below the horizontal through A.]

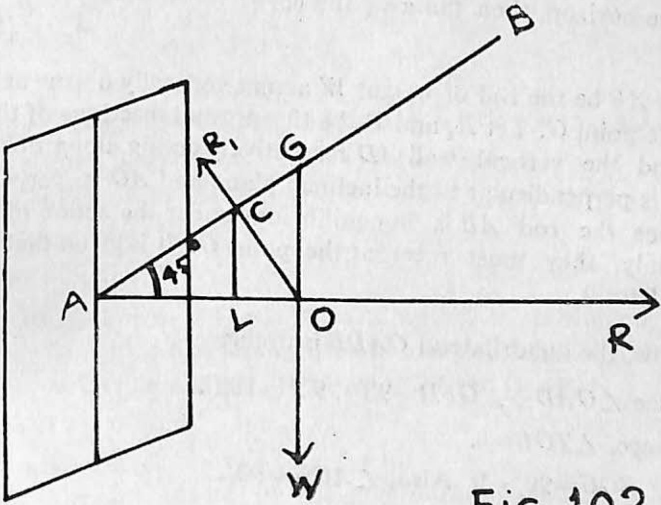

FIG. 102

Now, $\angle OAC = \angle AOC = 45°$.

Also, $\angle AGO = 45°$ and hence $\angle COG = 45°$. $\therefore AC = CO = CG$.

Now, if CL be drawn perpendicular to AO, then

$AL=LO$, since C is the middle point of AG.

Hence, $AL=AC\cos 45°=\frac{1}{2}AG\times\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}$.

Hence, the distance of the rail from the wall $=\frac{1}{2\sqrt{2}}$.

To find the reactions, R_1 and R, we apply Lami's theorem. By Lami's theorem, we have

$$\frac{R_1}{\sin 90°}=\frac{R}{\sin(90°+45°)}=\frac{W}{\sin(90°+45°)}$$

$$\Rightarrow \frac{R_1}{1}=\frac{R_2}{\cos 45°}=\frac{W}{\cos 45°}\Rightarrow R_1=\sqrt{2}R=\sqrt{2}\mathrm{W}$$

whence $R_1=\sqrt{2}\,W$ and $R=W$.

Ex. 15. A heavy uniform rod is in equilibrium with one end resting against a smooth vertical wall and the other against a smooth plane inclined to the wall at an angle θ. Prove that if α be the inclination of the rod to the horizon, then $\tan\alpha=\frac{1}{2}\tan\theta$.

[P.U. 1932]

Let AB be the rod of weight W acting vertically downwards through its middle point G. Let R_1 and R_2 be the normal reactions of the inclined plane and the vertical wall AD respectively acting along BO and AO so that BO is perpendicular to the inclined plane and AO is perpendicular to AD. Since the rod AB is in equilibrium under the action of these three forces only, they must meet at the point O. It is given that $\angle OAB=\alpha$ and $\angle ADB=\theta$.

Now, the quadrilateral $OADB$ is cyclic,

since $\angle OAD+\angle OBD=90°+90°=180°$.

Hence, $\angle TOB=\theta$.

$\therefore \angle BOG=90°-\theta$. Also, $\angle AOG=90°$.

Now, $\angle OGB=\alpha+\angle AOG=90°+\alpha$.

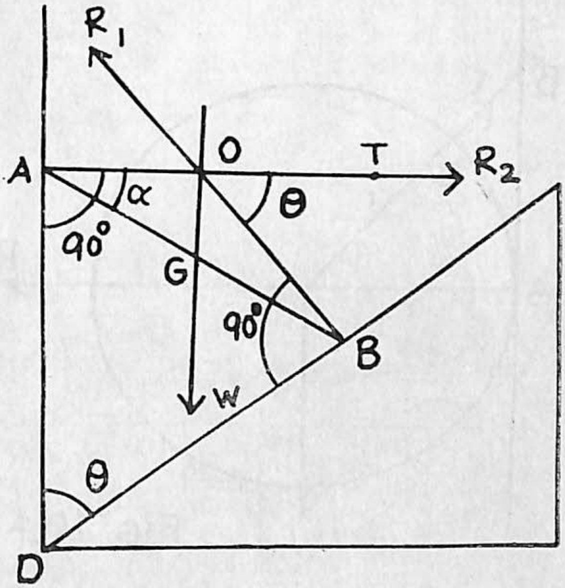

FIG. 103

Hence, using the cotangent formula in the triangle OAB, we have

$(1+1)\cot \angle OGB=\cot \angle AOG-\cot \angle GOB$

$\Rightarrow 2\cot (90^\circ+\alpha)=\cot 90^\circ-\cot (90^\circ-\theta)$

$\Rightarrow -2\tan \alpha=-\tan \theta$

$\Rightarrow \tan \alpha=\frac{1}{2}\tan \theta.$

Ex. 16. A sphere of radius r and weight W rests against a smooth vertical wall, to which it is attached by a string of length l, fastened to a point on its surface. Show that the tension of the string is

$$\frac{W(l+r)}{\sqrt{l^2+2lr}}.$$

Let C be the centre of the sphere through which its weight W is acting vertically downwards. AB is the vertical wall. Since the wall is smooth, the reaction R of the wall on the sphere will be normal to the wall and hence it will pass through C. BD is the string attached at a point D of the sphere. Since the two forces R and W meet at C, hence the tension T of the string will also pass through C. Given that $BD=l$ and $CD=r$. Applying Lami's theorem, we get

$$\frac{W}{\sin (180^\circ-\angle ACB)}=\frac{T}{\sin 90^\circ}\Rightarrow \frac{W}{\sin \angle ACB}=T.$$

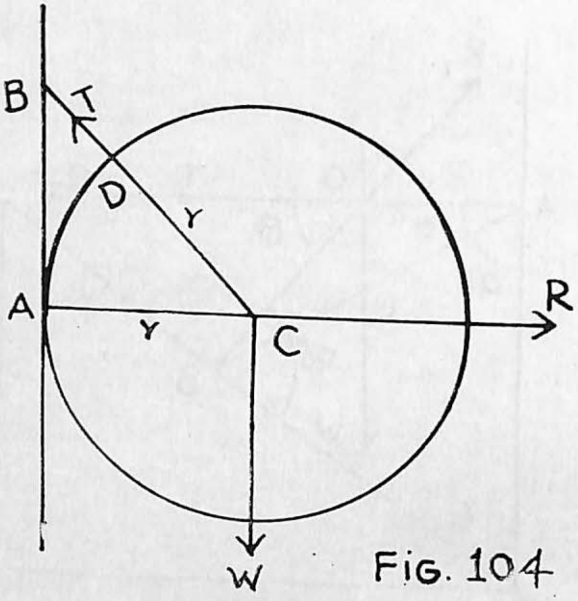

FIG. 104

Now, $\cos \angle ACB = \frac{r}{l+r}$ $\therefore$ $\sin \angle ACB = \frac{\sqrt{l^2+2lr}}{r+l}$.

Hence, $T = \frac{W(l+r)}{\sqrt{l^2+2lr}}$.

Ex. 17. A beam whose centre of gravity divides it into two portions a and b is placed inside a smooth sphere. Show that if θ be its inclination with the horizon in the position of equilibrium and 2α be the angle subtended by the beam at the centre of the sphere,

$\tan\theta = \frac{b-a}{b+a}\tan\alpha$.

[C.U. 1924]

[The centre of gravity of a body is the point at which its weight may be assumed to act.]

Let AB be the beam of weight W acting through G vertically downwards such that $AG=a$ and $BG=b$. The ends of the beam rest on the sphere at A and B. The normal reactions at A and B will pass through the centre O of the sphere. Hence, the line of action of the weight W will pass through O for equilibrium. AB makes an angle θ with the horizontal AMD so that $\angle BAD=\theta$. Let $OC \perp AB$.

Hence, C is the middle point of AB.

Now, $\angle COG = 90° - \angle OGC = 90° - \angle AGM = \angle GAM = \theta$.

Also, $GC = AC - AG = \frac{1}{2}(a+b) - a = \frac{1}{2}(b-a)$ and
$OC = AC \cot \angle AOC = \frac{1}{2}(a+b) \cot \alpha$ $[\because \angle AOC = \frac{1}{2} \angle AOB = \alpha.]$

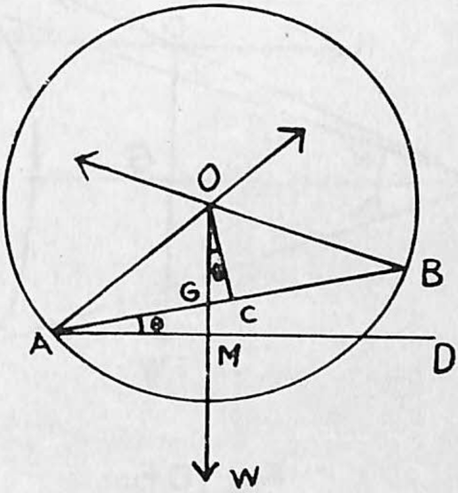

FIG. 105

Hence, from $\triangle GOC$,

$$\tan\theta = \frac{GC}{OC} = \frac{\frac{1}{2}(b-a)}{\frac{1}{2}(a+b)\cot\alpha} = \frac{b-a}{b+a}\tan\alpha.$$

Ex. 18. A solid cone of height h and semi-vertical angle α is placed with its base against a smooth vertical wall and is supported by a string of length l attached to the vertex and to a point in the wall. Show that the greatest possible length of the string is $h\sqrt{1+\frac{16}{9}\tan^2\alpha}$.

[Lucknow 1957]

[The centre of gravity of a solid right circular cone is on the axis at a distance $\frac{3}{4}$ h from the vertex.]

Let A be the vertex of the cone, BC, the base in contact with the wall, and X, the centre of the base so that AX is the altitude. Given that $AX = h$, $AP = l$ and $\angle BAX = \angle CAX = \alpha$ so that

$BX = CX = h \tan \alpha$.

The weight W of the cone will pass through G vertically downwards so that $GX = \frac{1}{4}h$. Let AP be the string attached to a point P of the vertical wall PC. Let T be the tension of the string. The reaction of the smooth wall on the cone will be normal to the wall. Since the cone

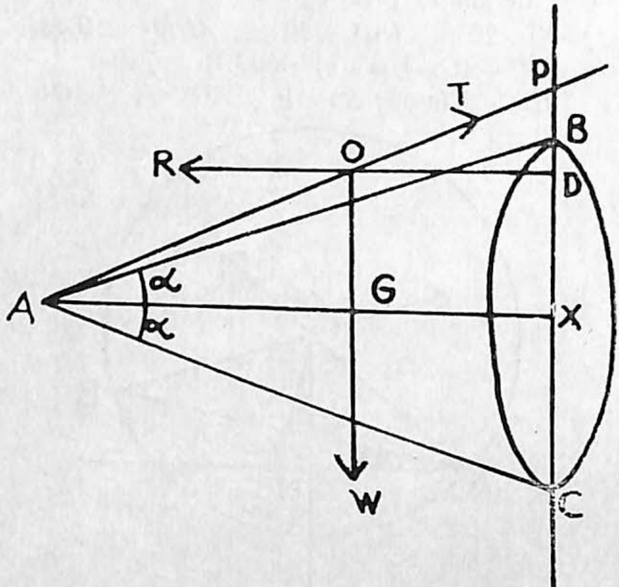

FIG. 106

is in equilibrium under these three forces, they must meet at a point, say O. OD is drawn perpendicular to the wall so that the normal reaction R will pass through O.

Here D is some point within the base BC. Hence, $DX \leqslant BX$.

$\Rightarrow OG \leqslant \text{h} \tan \alpha$

[$\because OGXD$ is a rectangle]. ...(1)

Now, $\triangle OAG$ and $\triangle PAX$ are similar.

$$\therefore \frac{OG}{PX} = \frac{AG}{AX} = \frac{3}{4} \Rightarrow OG = \frac{3}{4} \times PX = \frac{3}{4}\sqrt{AP^2 - AX^2} = \frac{3}{4}\sqrt{l^2 - \text{h}^2}.$$

Hence, from (1), we have $\frac{3}{4}\sqrt{l^2 - \text{h}^2} \leqslant \text{h} \tan \alpha$

$$\Rightarrow \frac{9}{16}(l^2 - \text{h}^2) \leqslant \text{h}^2 \tan^2 \alpha \quad \left[\because \text{each term on both sides of the inequality is positive}\right]$$

$$\Rightarrow l^2 \leqslant \text{h}^2\left(1 + \frac{16}{9}\tan^2 \alpha\right)$$

$$\Rightarrow l \leqslant \text{h}\sqrt{1 + \frac{16}{9}\tan^2 \alpha} \quad \left[\because l \text{ is always positive}\right].$$

Hence, the greatest possible length of the string is

$$\text{h}\sqrt{1 + \frac{16}{9}\tan^2 \alpha}.$$

Ex. 19. A fixed smooth hemispherical bowl of radius a is placed with its axis vertical, and a uniform rod of length l rests with one end inside the bowl and the other projecting over the rim. Prove that the length of the road outside the bowl is $\frac{1}{8}\left(7l-\sqrt{l^2+128a^2}\right)$, and hence deduce the shortest length of a rod that can rest in this manner.

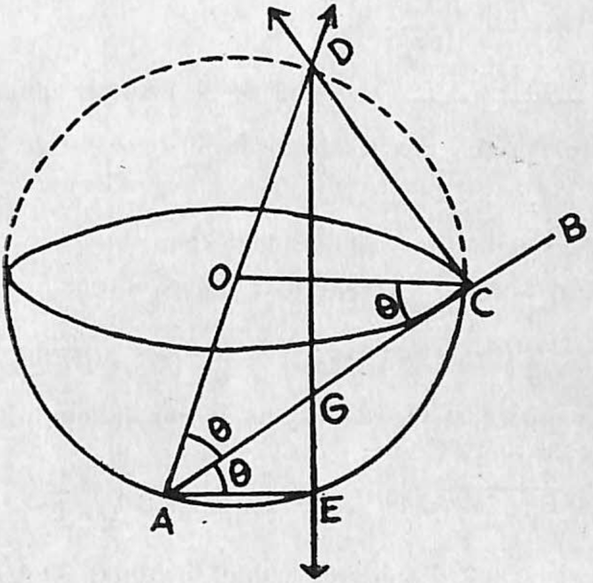

FIG. 107

Let AB be the rod, G its middle point through which the weight of the rod acts vertically downwards. A and C are the points of contact of the rod wirh the hemisphere. The reaction of the smooth bowl at A will be normal to the hemisphere at A and hence the reaction at A will pass through O, the centre of the bowl. The reaction at C of the rim on the rod will be normal to the rod and the rim. Since the rod is in equilibrium under the action of three forces only, they must meet at a point D. Let DG produced intersect the hemisphere at E. We join AE. Now, since AD is a diameter of the sphere and $\angle ACD=90°$, the point D must be the other extremity of the diameter AD. Hence, $\angle AED=90°$ and so AE is horizontal. Let θ be the inclination of the rod to the horizon.

$\therefore\ \angle EAC=\theta$. Also $\angle OAC=\angle OCA=\angle EAC$ [$\because OA=OC$ and $OC \parallel AE$]

Now, $OA=OC=a$ and $AB=l$.

From $\triangle ADE$, we have $AE=AD\cos 2\theta=2a\cos 2\theta$.

Also, from $\triangle AGE$, $AE=AG\cos\theta=\frac{l}{2}\cos\theta$.

Hence, $2a\cos 2\theta=\frac{l}{2}\cos\theta \Rightarrow 8a\cos^2\theta-l\cos\theta-4a=0$.

$$\Rightarrow \cos\theta=\frac{l\pm\sqrt{l^2+128a^2}}{16a}.$$

Now, $\frac{l-\sqrt{l^2+128a^2}}{16a}<0$ and so θ becomes obtuse which is absurd. Hence, rejecting this negative value of $\cos\theta$, we get.

$$\cos\theta=\frac{l+\sqrt{l^2+128a^2}}{16a}.$$

Now, the length of the rod outside the bowl

$$=BC=AB-AC=l-AD\cos\theta=l-2a\cos\theta$$

$$=l-\frac{2a}{16a}\left(l+\sqrt{l^2+128a^2}\right)=\frac{1}{8}\left(7l-\sqrt{l^2+128a^2}\right.\ .$$

Now, in order that the rod may be in equilibrium with its end B just touching the rim, $BC\geqslant 0$

$$\Rightarrow 7l\geqslant\sqrt{l^2+128a^2}\Rightarrow 49l^2\geqslant l^2+128a^2\Rightarrow l\geqslant 2a\sqrt{\frac{2}{3}}\ .$$

Hence, the required shortest length of the rod is $2a\sqrt{\frac{2}{3}}$.

Ex. 20. A smooth rod of length $2a$ has one end resting on a smooth plane of inclination α to the horizon and is supported by a horizontal rail which is parallel to the plane at a distance c from it. Show that the inclination θ of the rod to the incilned plane is given by the equation

$c\sin\alpha=a\sin^2\theta\cos(\theta-\alpha)$ [Allahabad 1939, 1957; Lucknow, 1953]

Let AB be the rod and G be its middle point through which its weight W acts vertically downwards. The end A of the rod rests on an inclined plane of inclination α. Since the plane is smooth, its reaction on the end A of the rod will be along AO, the normal to the inclined plane QR. The point M of the rod is its point of contact with the rail. (The rail may be considered as a horizontal straight line through M parallel to the plane QR, which is not shown in the figure). Since the distance of the rail from the plane is given to be c, the length of perpendicular from the point of contact M to the inclined plane QR must be c. Let MN be the length of

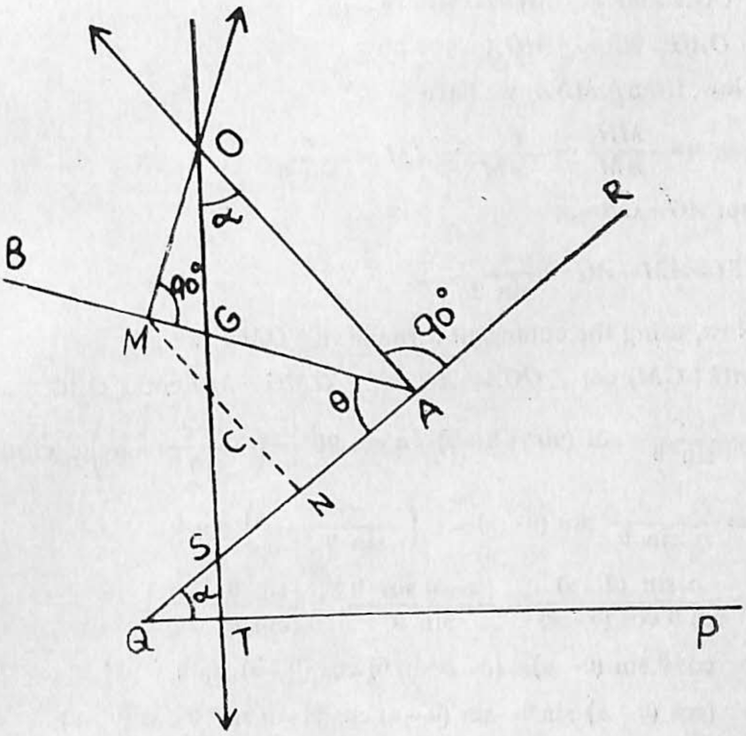

Fig. 108

this perpendicular so that $MN=c$. The reaction of the rail on the rod will be perdendicular to the rod. Let this reaction act along MO so that $\angle OMA=90°$. Since the rod is in equilibrium under the action of these forces only, they must meet at a point, say O. It is given that

$$\angle MAQ=\theta \text{ and } \angle AQP=\alpha.$$

Now, $\angle OAM=90°-\theta$.

Let the vertical through G intersect QR at S and the horizontal QP at T.

$$\text{Then } \angle GOA=90°-\angle OSA=90°-\angle QST$$

$$=90°-(90°-\alpha)=\alpha$$

$$\angle MOA=90°-\angle OAM=90°-(90°-\theta)=\theta.$$

$$\therefore \quad \angle MOG=\angle MOA-\angle GOA=\theta-\alpha.$$

$\therefore \quad \angle OGA = 90° + \angle MOG = 90° + \theta - \alpha$

$\angle OAG = 90° - \angle MOA = 90° - \theta.$

Now, from $\triangle MNA$, we have

$$\sin\theta = \frac{MN}{AM} = \frac{c}{AM} \therefore AM = \frac{c}{\sin\theta}$$

But $AG = GB = a$.

$$\therefore \quad GM = AM - AG = \frac{c}{\sin\theta} - a.$$

Now, using the cotangent formula in $\triangle OMA$, we get

$(AG + GM) \cot \angle OGA = AG \cot \angle OMG - MG \cot \angle OAG$

$$\Rightarrow \frac{c}{\sin\theta} \cot(90° + \theta - \alpha) = a \cot 90° - \left(\frac{c}{\sin\theta} - a\right) \cot(90° - \theta)$$

$$\Rightarrow -\frac{c}{\sin\theta} \tan(\theta - \alpha) = -\left(\frac{c}{\sin\theta} - a\right) \tan\theta$$

$$\Rightarrow \frac{c \sin(\theta - \alpha)}{\sin\theta \cos(\theta - \alpha)} = \frac{c - a\sin\theta}{\sin\theta} \times \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow c \cos\theta \sin(\theta - \alpha) = (c - a\sin\theta)\cos(\theta - \alpha)\sin\theta$$

$$\Rightarrow c\,[\cos(\theta - \alpha)\sin\theta - \sin(\theta - \alpha)\cos\theta] = a\sin^2\theta\cos(\theta - \alpha)$$

$$\Rightarrow c \sin\alpha = a \sin^2\theta \cos(\theta - \alpha).$$

Hence the result.

Ex. 21. A uniform rod 4m long is free to turn in a vertical plane about its upper end which is hinged. To a point of the rod 3m from the hinge is attached a string which, running perpendicular to the rod, passes over a pulley and supports a weight P. The rod is in equilibrium at an angle of 60° to the horizontal. Prove that P is one-third of the weight of the rod, and the reaction at the hinge makes an angle $\tan^{-1}\left(\frac{5\sqrt{3}}{3}\right)$ with the horizontal. Find also the reaction at the hinge.

Let AB be the rod hinged at A and G, its middle point through which its weight W acts vertically downwards. C is the point where the string CT is attached so that $\angle TCA = 90°$. M is the pulley over which the string passes. The weight P is attached at the extremity of the string so that the tension of the string CT is P acting along CT. The weight W of the rod and the tension P of the string meet at O. Hence, the reaction R of the hinge at A must pass through O along OA for equilibrium of the rod.

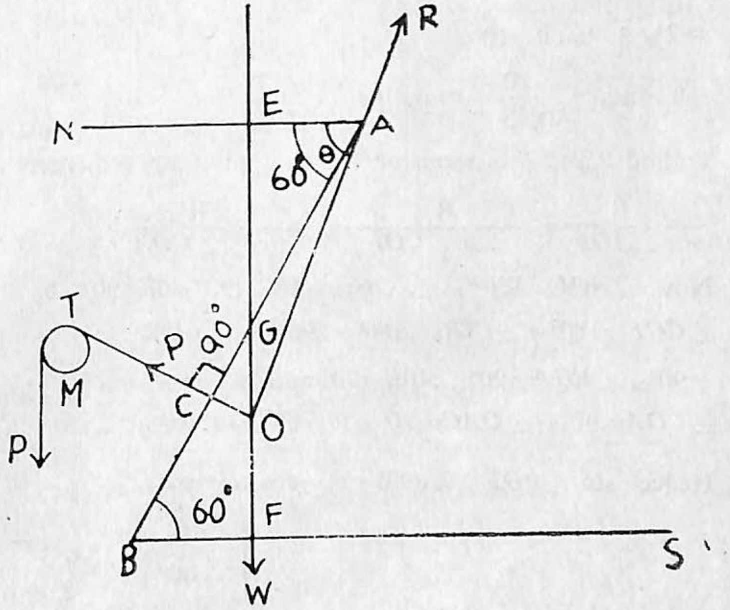

FIG 109

Let BS and AN be horizontal lines through B and A respectively. Let the vertical through G intersect AN and BS at E and F respectively [fig. 109]. It is given that $AC=3$m, $AB=4$m and $\angle ABS=60°$.

Hence, $\angle NAB=60°$. Let $\angle OAE=\theta$.

Now, $\angle OGC=\angle EGA=90°-60°=30°$,

$\angle OCG=90°$, $\angle OAG=\theta-60°$,

$AG=2$m, $GC=AC-AG=(3-2)$ m$=1$m.

Hence, using the cotangent formula in $\triangle COA$, we have

$$(AG+GC)\cot\angle OGC=GC\cot\angle GOA-AG\cot\angle OCG$$

$$\Rightarrow 3\cot 30°=\cot(\theta-60°)-2\cot 90°$$

$$\Rightarrow 3\sqrt{3}=\cot(\theta-60°)=\frac{1+\tan\theta\tan 60°}{\tan\theta-\tan 60°}$$

$$=\frac{1+\sqrt{3}\tan\theta}{\tan\theta-\sqrt{3}}$$

$$\Rightarrow 3\sqrt{3}\ \tan\theta - 9 = 1 + \sqrt{3}\ \tan\theta$$

$$\Rightarrow 2\sqrt{3}\ \tan\theta = 10$$

$$\Rightarrow \theta = \tan^{-1}\frac{10}{2\sqrt{3}} - \tan^{-1}\left(\frac{5}{3}\sqrt{3}\right).$$

To find P and R in terms of W, we apply Lami's theorem and get

$$\frac{P}{\sin\angle AOF} = \frac{R}{\sin\angle COF} = \frac{W}{\sin\angle COA}. \qquad \ldots (A)$$

Now, $\angle AOF = 180° - \angle AOG = 180° - (90° - \theta) = 90° + \theta$,

$\angle COF = 180° - \angle COG = 180° - (90° - \angle CGO)$.

$= 90° + \angle EGA = 90° + 30° = 120°$ and

$\angle COA = 90° - \angle OAC = 90° - (\theta - 60°) = 150° - \theta$.

Hence, $\sin\angle AOF = \sin(90° + \theta) = \cos\theta = \dfrac{3}{\sqrt{84}}$,

$$\left[\because \tan\theta = \frac{5\sqrt{3}}{3}\right]$$

$\sin\angle COF = \sin\ 120° = \dfrac{\sqrt{3}}{2}$,

$$\sin\angle\ COA = \sin\ (150° - \theta) = \frac{1}{2}\ \cos\ \theta + \frac{\sqrt{3}}{2}\ \sin\ \theta$$

$$= \frac{1}{2} \times \frac{3}{\sqrt{84}} + \frac{\sqrt{3}}{2} \times \frac{5\sqrt{3}}{\sqrt{84}}$$

$$= \frac{18}{2\sqrt{84}} - \frac{9}{\sqrt{84}}.$$

Hence, from (A),

$$\frac{P}{\frac{3}{\sqrt{84}}} = \frac{R}{\frac{\sqrt{3}}{2}} = \frac{W}{\frac{9}{\sqrt{84}}}$$

$$\Rightarrow P = \frac{W}{3} \text{ and } R = \frac{\sqrt{7}}{3}\ W.$$

(c) Equilibrium of any number of forces:

Ex. 22. A bar AB of weight $40N$ and length 8 m is hinged at A to a point in a vertical wall, and is maintained in a horizontal position by means of a chain attached to *B* and to a point in the wall 5m above *A*. If the bar carries a load of 20*N* at a point 6m from *A*, calculate the tension in the chain, and the magnitude and direction of the reaction at *A* between the bar and the wall.

Let *AB* be the horizontal rod higned at *A* to the vertical wall *AC*. The weight 40 *N* of the rod acts at *B*, the middle point of *AB*, vertically downwards. *BC* is the string attached to the end *B* of the rod and a point *C* of the wall such that *AC*=5m. Let *T* be the tension of the string inclined at an angle ϕ with the horizontal *BA* and let *R* be the reaction at the hinge *A*, making an angle θ with *AB*.

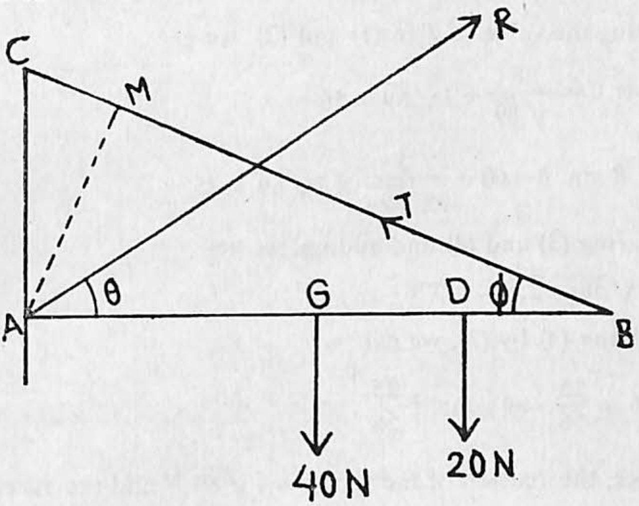

Fig. 110

Now, $BC=\sqrt{AC^2+AB^2}=\sqrt{5^2+8^2}=\sqrt{89}$

$$\therefore \sin\phi=\frac{5}{\sqrt{89}} \text{ and } \cos\phi=\frac{8}{\sqrt{89}}.$$

Resolving all the forces horizontally and vertically, we get [cf. section 7.10 (i) and (ii)]

$$R\cos\theta=T\cos\phi=\frac{8T}{\sqrt{89}} \qquad \ldots(1)$$

$$\text{and } R\sin\theta+T\sin\phi=20+40=60$$

$$\Rightarrow R\sin\theta=60-\frac{5T}{\sqrt{89}}. \qquad \ldots(2)$$

Let *AM* be perpendicular to *BC*. Then

$$AM=AB\sin\phi=8\times\frac{5}{\sqrt{89}}=\frac{40}{\sqrt{89}}.$$

Taking moments of all the forces about A, we get [cf. section 7.10 (iii)]

$$T.8 \sin \phi - 40 \times 4 - 20 \times 6 = 0$$

$$\Rightarrow \quad 8T \times \frac{5}{\sqrt{89}} = 280 \Rightarrow T = 7\sqrt{89} \,.$$

Putting the value of T in (1) and (2), we get

$$R \cos \theta = \frac{8}{\sqrt{89}} \times 7\sqrt{89} = 56 \qquad \ldots (3)$$

$$\text{and } R \sin \theta = 60 - \frac{5}{\sqrt{89}} \times 7\sqrt{89} = 25. \qquad (4)$$

Squaring (3) and (4) and adding, we get

$$R - \sqrt{56^2 + 25^2} = \sqrt{3761} \,.$$

Dividing (4) by (3), we get

$$\tan \theta = \frac{25}{56} \Rightarrow \theta = \tan^{-1} \frac{25}{56} \,.$$

Hence, the tension of the string is $7\sqrt{89}\, N$ and the reaction at the hinge is $\sqrt{3761}\, N$, making an angle $\tan^{-1} \frac{25}{56}$ with the horizon.

Ex. 23. A ladder of length $2\,l$ and weight W rests against a smooth vertical wall. Its lower end is in contact with the floor which is smooth and is prevented from slipping by a string of length a connecting it with the junction of the wall and the floor. If a person of weight $2W$ stands on the rung of the ladder distant $\frac{1}{2}l$ from its lower end, determine the reactions at the two ends of the ladder and the tension of the string.

[C.U. 1941]

Let AB be the ladder of length $2l$ and G be its middle point through which its weight acts vertically downwards. The end B of the ladder rests on the vertical wall OB. OA is the horizontal string connected with the end A of the ladder and the junction O of the floor and the wall. C is the position of the man on the ladder and his weight $2W$ acts vertically downwards through C. It is given that $AC = \frac{1}{2}l$ and $AB = 2l$. Let T be the tension of the string AO and R_1, R_2 be the reactions of the floor and the wall respectively. Since the floor and the wall are smooth, the direction of R_1 and R_2 will be normal to AO and OB respectively. Let the lines of action of W and $2W$ intersect OA at E and D respectively [fig. 111].

Let $\angle OAB=\theta$.

$$\therefore \cos\theta = \frac{OA}{AB}=\frac{a}{2l}$$

and $\sin\theta=\dfrac{\sqrt{4l^2-a^2}}{2l}$.

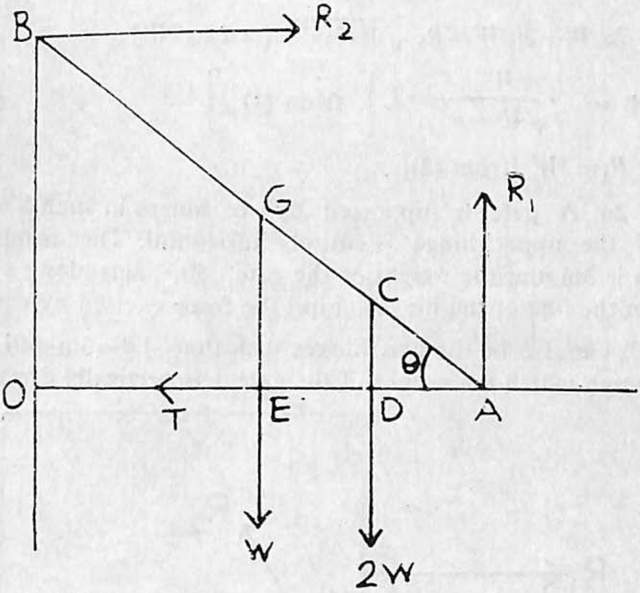

FIG. 111

Resolving all the forces horizontally and vertically, we get

$$T=R_2 \qquad \ldots(1)$$

and $R_1=3W$. ...(2)

Taking moments of all the forces about O, we get

$$-W.\,OE-2W.OD-R_2.DB+R_1.AO=0. \qquad \ldots(3)$$

Now, $OE=\frac{1}{2}\,.\,OA=\frac{a}{2}$.

$$OD=AO-AD=a-\frac{l}{2}\cos\theta=a-\frac{l}{2}\cdot\frac{a}{2l}=\frac{3a}{4},$$

$OB=\sqrt{AB^2-OA^2}=\sqrt{4l^2-a^2}$ and

$AO=a.$

Hence, from (3), we get

$$-W.\ \frac{a}{2}-2W.\ \frac{3a}{4}-R_2.\ \sqrt{4l^2-a^2}+R_1.a=0$$

$$\Rightarrow -2a\ W+3a\ W=R_2\ \sqrt{4l^2-a^2}\quad [\because R_1=3W]$$

$$\Rightarrow R_2=\frac{aW}{\sqrt{4l^2-a^2}}=T\left[\text{ from (1)}\right]$$

and $R_1=3W$ [from (2)].

Ex. 24. A gate is supported by two hinges in such a way that the action of the upper hinge is entirely horizontal. The distance between the hinges is 3m and the weight of the gate, 60*N*, acts along a vertical line $3\frac{1}{2}$ m from the line of the hinges. Find the force exerted by each hinge.

Let *A* and *B* be the two hinges such that *AB*=3m and let *G* be the point through which the weight of the gate acts vertically downwards. The

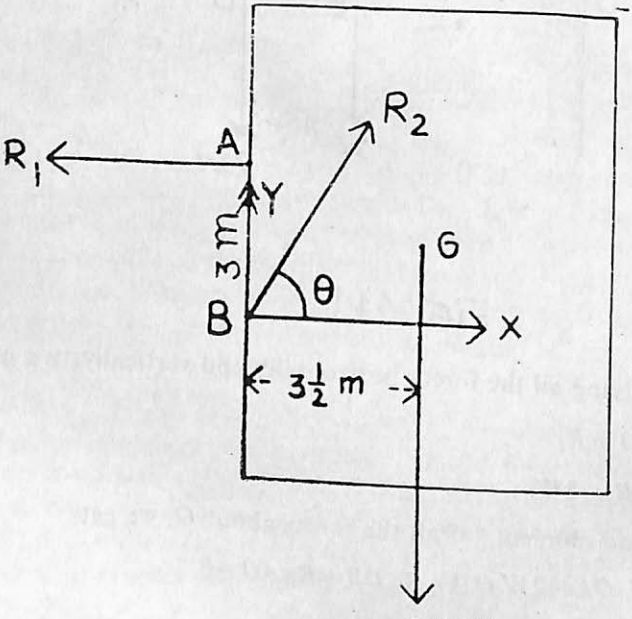

FIG. 112

line of action of this weight is given to be at a distance of $3\frac{1}{2}$ m from AB. Let R_1 be the reaction of the upper hinge, which is given to be horizontal. Let R_2 be the reaction of lower hinge [fig. 112] making an angle θ with the horizontal and let X and Y be its horizonial and vertical components. Resolving all the forces horizontally and vertically, we get

$$R_1 = X \qquad \text{...(1)}$$

and $Y = 60.$ ...(2)

Taking moments about B, we get

$$60 \times \frac{7}{2} = R_1 \times 3 \Rightarrow R_1 = 70.$$

$\therefore$ From (1), $X = 70.$

Hence, the reaction at the lower hinge$=\sqrt{X^2+Y^2}=\sqrt{70^2+60^2}N$ $=10\sqrt{85}\ N$ and that at the upper hinge is $70N$.

Ex. 25. Forces, P, Q and R act along the sides BC, CA and AB respectively of a triangle ABC and forces P', Q' and R' act along AG, BG and CG respectively, where G is the centroid of the triangle ABC. If the six forces are in equilibrium, prove that

$$\frac{PP'}{AG.BC} + \frac{QQ'}{BG.CA} + \frac{RR'}{CG.AB} = 0.$$

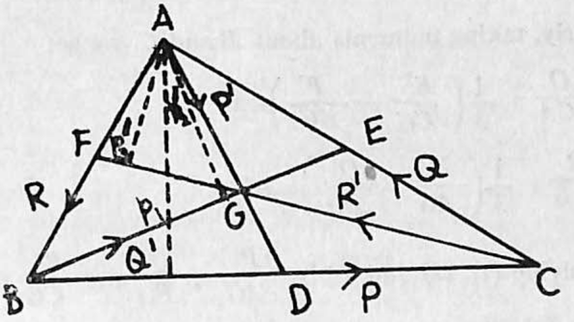

FiG. 113

Le the medians AD, BE and CF intersect at G, the centroid of the triangle ABC. Let p_1 be the length of the perpendicular from A to BC and $\triangle$ be the area of $\triangle ABC$.

Then $\frac{1}{2} BC.\ p_1 = \triangle \Rightarrow p_1 = \frac{2\triangle}{BC}$.

If p_2' be the length of the perpendicular from A to BE, then

$\frac{1}{2}BE.\,p_2'$ = area of $\triangle ABE = \frac{1}{2}\triangle$

$$\Rightarrow p_2' = \frac{\triangle}{BE} = \frac{\triangle}{\frac{3}{2}\,BG} \quad [\because BG:GE=2:1]$$

$$= \frac{2}{3}\cdot\frac{\triangle}{BG}.$$

Similarly, if p_3' be the length of the perpendicular from A to CF, then we have

$$p_3' = \frac{2}{3}\cdot\frac{\triangle}{CG}.$$

Now, since the six forces are in equilibrium, hence taking moments about A, we have

$$Pp_1 + Q'p'_2 - R'p_3' = 0$$

$$\Rightarrow P.\,\frac{2\triangle}{BC} + Q'.\,\frac{2}{3}.\,\frac{\triangle}{BG} - R'.\,\frac{2}{3}.\,\frac{\triangle}{CG} = 0$$

$$\Rightarrow \frac{P}{BC} + \frac{1}{3}\left(\frac{Q'}{BG} - \frac{R'}{CG}\right) = 0 \qquad \ldots(1)$$

Similarly, taking moments about B and C, we get

$$\frac{Q}{CA} + \frac{1}{3}\left(\frac{R'}{CG} - \frac{P'}{AG}\right) = 0 \qquad \ldots(2)$$

$$\text{and } \frac{R}{AB} + \frac{1}{3}\left(\frac{P'}{AG} - \frac{Q'}{BG}\right) = 0. \qquad \ldots(3)$$

Multiplying (1), (2) and (3) by $\frac{P'}{AG}$, $\frac{Q'}{BG}$ and $\frac{R'}{CG}$ respectively and adding, we get

$$\frac{PP'}{AG.BC} + \frac{QQ'}{BG.CA} + \frac{RR'}{CG.AB} + \frac{1}{3}\left(\frac{P'Q'}{BG.AG} - \frac{P'R'}{CG.AG} + \frac{R'Q'}{CG.BG} - \frac{P'Q'}{AG.BG} + \frac{P'R'}{AG.CG} - \frac{Q'R'}{BG.CG}\right) = 0$$

$$\Rightarrow \frac{PP'}{AG.BC} + \frac{QQ'}{BG.CA} + \frac{RR'}{CG.AB} = 0.$$

Ex. 25. Forces P, Q, R and S act along the sides AC, BC, CD and DA of a quadrilateral which is inscribed in a circle whose centre is on AB. If they keep a rigid body in equilibrium, prove that

$$R^2=P^2+Q^2+S^2+\frac{2PQS}{R}.$$ [C.U. 1945]

Let O be the centre of the circle, lying on AB. Since the four forces are in equilibrium, the resultant of P and Q must be equal in magnitude to that of R and S.

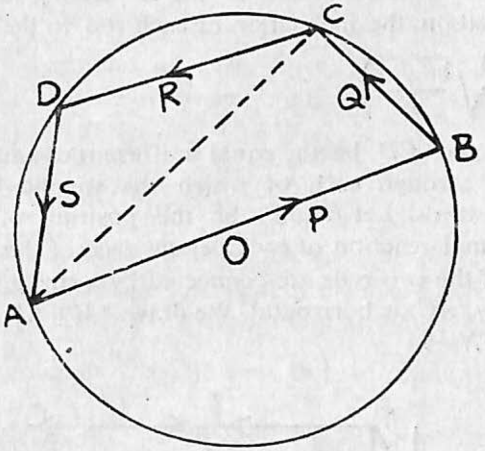

FIG. 114

Hence,

$$P^2+Q^2+2PQ\cos(180^\circ-B)=R^2+S^2+2RS\cos(180^\circ-D)$$
$$\Rightarrow P^2+Q^2-2PQ\cos B=R^2+S^2-2RS\cos B$$
$[\because \angle D+\angle B=180^\circ$, the quadrilateral being cyclic$]$
$$\Rightarrow P^2+Q^2-R^2-S^2-2(PQ+RS)\cos B=0. \qquad \text{..(1)}$$

Now, taking moments of the forces about B, we get

$$R.BC\sin\angle BCD+S.\ AB\sin\angle BAD=0$$
$$\Rightarrow R.BC\sin(180^\circ-\angle BAD)+S.\ AB\sin\angle BAD=0$$
$$\Rightarrow R.BC\sin\angle BAD+S.AB\sin\angle BAD=0$$
$$\Rightarrow R.AB\cos B+S.AB=0\quad [\because \angle ACB=90^\circ]$$
$$\Rightarrow \cos B=-\frac{S}{R}. \qquad \text{...(2)}$$

Hence, from (1) and (2), we have

$$P^2+Q^2-R^2-S^2-2(PQ+RS)\left(-\frac{S}{R}\right)=0$$

$$\Rightarrow P^2+Q^2-R^2-S^2+2.\,\frac{PQS}{R}+2S^2=0$$

$$\Rightarrow R^2=P^2+Q^2+S^2+2.\,\frac{PQS}{R}.$$

Ex. 26. Two equal uniform heavy rods are connected at one extremity by a thin string and the system is placed symmetrically in a vertical plane with the rods resting on two smooth pegs in the same horizontal line. If a be the length of each rod, b the distance between the pegs and (l<b) is the length of the connecting string, show that in equilibrium position, the inclination of each rod to the horizon is

$$\cos^{-1}\left(\sqrt[3]{\frac{b-l}{a}}\right).$$

Let AB and CD be the equal uniform rods and G_1 and G_2 be their middle points through each of which the weight W of each rod acts vertically downwards. Let E and F be the position of the two pegs and R be the normal reaction of each peg on each of the rods. The extremities A and C of the two rods are connected by a string having tension T. From symmetry, AC is horizontal. We draw $AM \perp EF$, $CN \perp EF$ and let $\angle AEM = \angle CFN = \theta$.

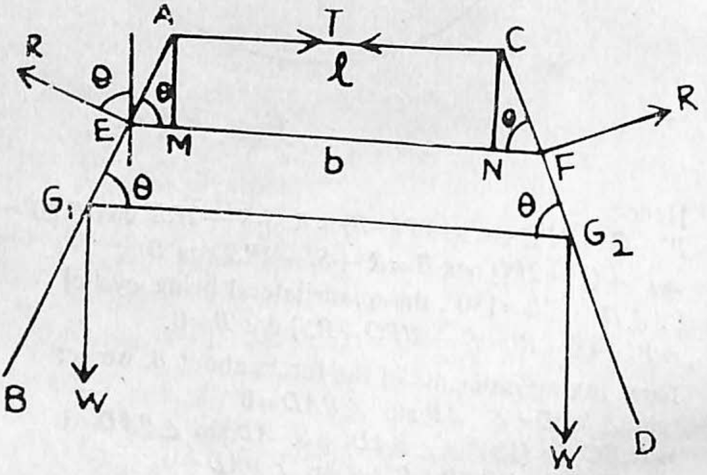

FIG 115

It is given that

$AB=CD=$a, $EF=$b and $AC=l$.

We can consider the equilibrium of any one of the rods due to the symmetry of the system.

Considering the equilibrium of the rod AB and resolving vertically, we get

$R \cos \theta = W$. ...(1)

Now, $EM = EF - MF = EF - (MN + NF) = EF - MN - EM$
$= \mathrm{b} - l - EM$

$\Rightarrow 2EM = \mathrm{b} - l \Rightarrow EM = \frac{1}{2}(\mathrm{b} - l)$.

Hence, $AE = EM \sec \theta = \frac{1}{2}(\mathrm{b} - l) \sec \theta$.

Now, taking moments of the forces acting on AB about A, we get
$WAG_1 \cos \theta = R.AE = R.\frac{1}{2}(\mathrm{b} - l) \sec \theta \Rightarrow R(\mathrm{b} - l) \sec^2\theta = W\mathrm{a}$. ...(2)

Dividing (1) by (2), we get

$$\frac{\cos^3 \theta}{\mathrm{b} - l} = \frac{1}{\mathrm{a}} \Rightarrow Q = \cos^{-1} \sqrt[3]{\frac{\mathrm{b} - l}{\mathrm{a}}}\ .$$

Ex. 27. Two equal smooth spheres, each of weight W and radius r are placed within a thin hollow vertical cylinder of radius a ($<$2r), open at both ends, and resting on a horizontal table. Prove that the least weight of the cylinder so that it may not be upset is $2W\left(1 - \frac{\mathrm{r}}{\mathrm{a}}\right)$.

Let O_1 and O_2 be the centres of the two spheres each of radius r. The sum of the diameters of the two spheres is 4r and the diameter of the cylinder is 2a. Since 2a $<$4r, hence the spheres when placed within the cylinder will rest one above the other in an inclined position as shown in the figure [fig. 116]. The lower sphere with centre O_1 is in contact with the horizontal table and let A be the point of contact of this sphere with the table. Both the spheres are in contact with the open cylinder as as well as with each other. Let B and C be the points of contact of the two spheres with the cylinder. The lower sphere is acted upon by its weight W acting vertically downwards through its centre O_1, the normal reaction R of the table on this sphere, acting through O_1, the sideways normal reaction of the cylinder at the point of contact B acting horizontally through O_1 and the normal reaction of the sphere with centre O_2 acting through O_1. Similarly, the sphere with centre O_2 will be acted on by its weight W acting vertically downwards through its centre O_2' the sideways normal reaction of the cylinder at the point of contact C acting horizontally through O_2 and the normal reaction of the lower sphere acting through O_2. The mutual action and reaction of the two spheres must be equal in magnitude but opposite in sige and hence they will balance each other. Considering the equilibrium of the whole system together and resolving the forces vertically, we get $R = 2W$.

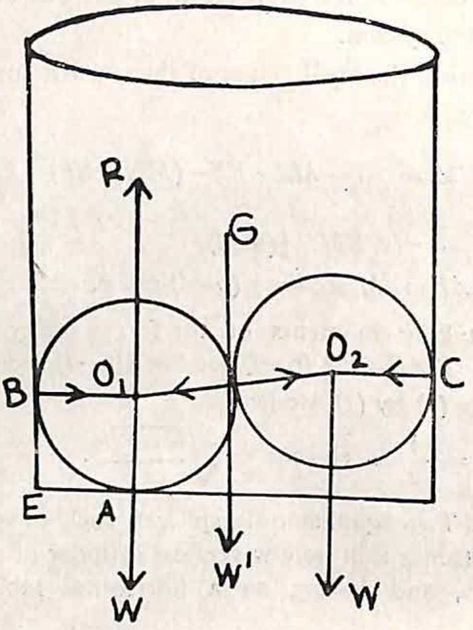

FiG. 116

Hence, the resultant vertical force at O_1 is $2W-W=W$ acting in the vertically upward direction. This force together with the vertically downward force W acting through O_2 will form a couple of moment

$W\times$ the horizontal distance between O_1 and $O_2=W\times 2(a-r)$. The horizontal reactions of the cylinder on the spheres will, therefore, form a couple neutralising the above couple. Hence, the sideways pressures on the cylinder exerted by the spheres at their points of contact with it must be equivalent to a couple of moment $2W(a-r)$ and this tends to turn about an extremity, say E, of the base. If W' be the weight of the cylinder, then its moment about E is $W'a$.

(The weight of a cylinder acts vertically downwards along its axis). This moment tends the cylinder to rotate in the opposite direction so as to prevent the cylinder to become upset about E. Thus in order that the cylinder may not be upset

$$W'a \leqslant 2W(a-r)$$

$$\Rightarrow W' \geqslant 2W\left(1-\frac{r}{a}\right)$$

i. e. the least wdight of the cylinder must be

$2W\left(1-\frac{r}{a}\right)$ in order that it may not be upset.

Ex. 28. Inside a fixed hollow cylinder of radius b, whose axis is horizontal are placed symmetrically and longitudinally two equal smooth cylinders, each of radius a. A third cylinder of equal radius but double the weight is placed symmetrically on the first two, all the axes being parallel. Prove that the third cylinder will force the other two apart if $b > a\ (1+\sqrt{13})$.

Fig. 117 represents the vertical section of the four cylinders through the middle points of their equal axes. O_1, O_2, O_3 and O_4 are centres of the circular sections of the cylinders. Weight W, W and $2W$ of the three cylinders act through O_1, O_2 and O_3 respectively in the vertically downward direction. Let the mutual actions and reactions between the lower and the upper cylinders be R, R acting along $O_1 O_3$ and $O_2 O_3$ and those between the two lower cylinders be R'', R'' acting along $O_1 O_2$. Let R', R' be the reactions of the fixed cylinder on the two lower cylinders, $AO_1 O_4$ and $BO_2 O_4$, A and B being the points of contact of the fixed cylinder with the two lower cylinders.

Let $\angle O_1O_4O_3 = \angle O_3O_4O_2 = \theta$.

Considering the three cylinders as one body, we observe that the system is in equilibrium under R', R' acting along AO_4 and BO_4 and the three vertical weights W, W and $2W$ acting at O_1, O_2 and O_3 respectively. Resolving these forces vertically, we get

$$2R' \cos\theta = 4W \;\Rightarrow R' \cos\theta = 2W. \qquad \dots(1)$$

We now consider the equilibrium of the upper cylinder which is acted upon by the forces R, R along O_1O_3 and $O_2 O_3$ and the weight $2W$ acting at O_3 in the vertically downward direction. Let C be the point of contact of the two lower cylinders. Hence, resolving these forces vertically, we have

$$2R \cos \angle O_1O_3C = 2W. \qquad \dots(\text{A})$$

Now, since $O_1O_3 = O_1O_2 = O_2O_3 = 2a$

and $O_3C_2 \perp O_1O_2$, hence $\angle O_1O_3C = \angle CO_3O_2 = 30°$.

Hence, we have from (A),

$$2R \cos 30° = 2W \;\Rightarrow \sqrt{3}\, R = 2W. \qquad \dots(2)$$

We now consider the equilibrium of any one of the two lower cylinders. It is in equilibrium under the ferces W, R, R' and R''. Resolving these forces horizontally, we get

$$R'' + R \cos 60° = R' \sin\theta$$
$$\Rightarrow R'' = R' \sin\theta - \tfrac{1}{2}R. \qquad \dots(3)$$

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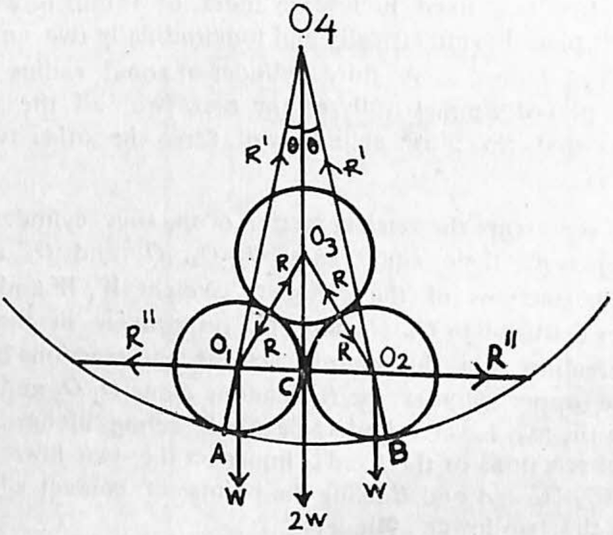


FiG. 117

Substituting the values of R' and R from (1) and (2) in (3), we get

$$R''=\frac{2W}{\cos\theta}\cdot\sin\theta-\frac{1}{2}\cdot\frac{2W}{\sqrt{3}}=2W\left(\tan\theta-\frac{1}{2\sqrt{3}}\right).$$

Now, the upper cylinder will force the two lower cylinders only when $R''<O$

i.e. when $\tan\theta<\frac{1}{2\sqrt{3}}$.

Now, $O_1O_4=\mathrm{b}-\mathrm{a}$, $O_1C=\mathrm{a}$.

Hence, $O_4C=\sqrt{(\mathrm{b}-\mathrm{a})^2-\mathrm{a}^2}=\sqrt{\mathrm{b}^2-2\mathrm{ab}}$

$\therefore \tan\theta=\frac{O_1C}{O_4C}=\frac{\mathrm{a}}{\sqrt{\mathrm{b}^2-2\mathrm{ab}}}$.

Hence, the upper cylinder will force the two lower cylinders

if $\quad\frac{\mathrm{a}}{\sqrt{\mathrm{b}^2-2\mathrm{ab}}}<\frac{1}{2\sqrt{3}}$.

i. e. if $\frac{\mathrm{a}^2}{\mathrm{b}^2-2\mathrm{ab}}<\frac{1}{12}$ $\left[\because \mathrm{b}^2-2\mathrm{ab}>0,\ \mathrm{b}\text{ being greater than }2\mathrm{a}\ .\right]$

i. e. if $12a^2 - b^2 + 2ab < 0$

i. e. if $b^2 - 2ab - 12a^2 > 0$

i. e. if $b > \dfrac{2a + \sqrt{4a^2 + 48a^2}}{2}$ $\Big[\because$ this implies that

$$b > \frac{2a - \sqrt{4a^2 + 48a^2}}{2} \Big]$$

i. e. if $b > a(1 + \sqrt{13})$.

Hence the result.

Ex. 29. A triangular lamina of weight W is supported by three vertical strings attached to its angular points so that the plane of the triangle is horizontal. A particle of weight W is placed on the triangle at its ortho-centre. Prove that if T_1, T_2 and T_2 be the tensions in the three strings, then

$$\frac{T_1}{1 + 3\cot B, \cot C} = \frac{T_2}{1 + 3\cot C. \cot A} = \frac{T_3}{1 + 3\cot A. \cot B}$$

[The weight of a triangular lamina acts at the centroid of the triangle in the vertically downward direction.]

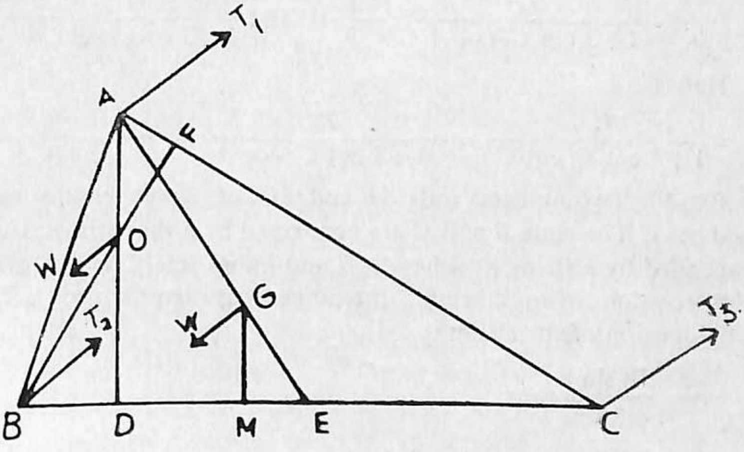


FiG. 118

Let ABC be the triangular lamina, $AD \perp BC$, AE is a median and G is the centroid of the triangle. Let O be the ortho-centre of $\triangle ABC$. Then weights W, W act at G and O in the vertically downward direction.

Let T_1, T_2 and T_3 be the tensions of the strings attached with A, B and C respectively. If $GM \perp BC$, then $GM = \frac{1}{3} AD$.

We join BO and produce it to meet AC at F. Then $\angle BFC = 90°$,

$$OD = BD \tan \angle OBD = BD \cot C \; [\because \angle BFC = 90°]$$

$$= AD \cot B . \cot C.$$

Now, the triangle is in equilibrium under the tensions T_1, T_2 and T_3 acting vertically upwards and the weights W, W acting at G and O in the vertically downward direction. Taking moments of all these vertical forces about the side BC which is horizontal, we get

$$T . AD = W . GM + W . OD$$

$$= W \frac{1}{3} AD + W . AD \cot B . \cot C$$

$$= \frac{1}{3} W . AD \, (1 + 3 \cot B \cot C)$$

$$\Rightarrow \frac{T_1}{1 + 3 \cot B . \cot C} = \frac{1}{3} W.$$

Similarly, taking moments of the forces about CA and AB, we get

$$\frac{T_2}{1 + 3 \cot C . \cot A} = \frac{1}{3} W \text{ and } \frac{T_3}{1 + 3 \cot A . \cot B} = \frac{1}{3} W$$

Hence,

$$\frac{T_1}{1 + 3 \cot B . \cot C} = \frac{T_2}{1 + 3 \cot C . \cot A} = \frac{T_3}{1 + 3 \cot A . \cot B} .$$

Ex. 30. Two uniform rods AB and AC of equal lengths are freely jointed at A. The ends B and C are connected by a thin string. The system is suspended by a string attached to B and hangs freely under gravity. If in the position of equilibrium, the angle between the rods is 2α, show that the tension of the string is

$$\frac{3W \sin \alpha}{\sqrt{1 + 3 \sin^2 \alpha}} ,$$

where $2W$ is the weight of either rod.

[Tranancore 1940]

Let E and F be the middle points of the rods AB and AC respectively. Hence, the weights of the rods act through E and F in the vertically downward direction. Let the tension of the vertical string attached at B be T' and that of the string connecting B and C be T. We join EF and draw

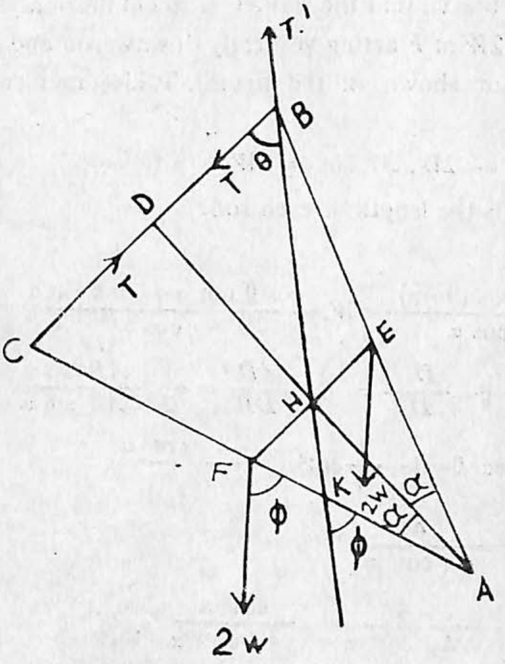


Fig. 119

$AD \perp BC$ cutting EF at its middle point H. The resultant of two equal like parallel forces $2W$, $2W$ at E and F will clearly pass through H in the vertically downward direction. If we consider the two rods as one body, then the body is in equilibrium under the forces T' in the vertically upward direction and the weight $4W$ of the rods acting through H in the vertically downward direction. Hence, these two forces must act along the same line so that the vertical drawn through B must pass through H. Let AC and BC make angels ϕ and θ respectively with this vertical line.

Now, $\angle HKA = 180° - \phi$, $\angle KHA = \angle DHB = 90° - \theta$ and $\angle KAH = \alpha$.

$\therefore$ In $\triangle HKA$,

$$\angle HKA + \angle KHA + \angle KAH = 180°$$

$$\Rightarrow \quad 180° - \phi + 90° - \theta + \alpha = 180°$$

$$\Rightarrow \quad \phi = 90° - (\theta - \alpha) \qquad \ldots\ldots\ldots(1)$$

We now observe that the rod AC is in equilibrium under the forces T at C along CB, $2W$ at F acting vertically downwards and the reaction at the point A (not shown in the figure). Taking moments of these forces about A, we get

$$T.\ 2a \cos\alpha = 2W.\ AF \sin\phi = 2Wa \cos(\theta-\alpha), \qquad \text{[from (1)]}$$

where $2a$ is the length of each rod.

Hence,

$$T = \frac{W\cos(\theta-\alpha)}{\cos\alpha} = W.\frac{\cos\theta\cos\alpha+\sin\theta\sin\alpha}{\cos\alpha} \qquad ...(2)$$

$$\text{Now, } \tan\theta = \frac{DH}{DB} = \frac{1}{2}\cdot\frac{AD}{DB} = \frac{1}{2}\cdot\frac{AB\cos\alpha}{AB\sin\alpha} = \frac{1}{2}\cot\alpha\ .$$

$$\text{Hence, } \sec^2\theta = 1+\frac{1}{4}\cot^2\alpha = \frac{4+\cot^2\alpha}{4}$$

$$\Rightarrow \cos^2\theta = \frac{4}{4+\cot^2\alpha}$$

$$\therefore\ \sin^2\theta = 1-\frac{4}{4+\cot^2\alpha} = \frac{\cot^2\alpha}{4+\cot^2\alpha}\ .$$

Hence, from (2),

$$T = W\left(\frac{2}{\sqrt{4+\cot^2\alpha}} + \frac{\cot\alpha\tan\alpha}{\sqrt{4+\cot^2\alpha}}\right)$$

$$= \frac{3W}{\sqrt{3+\text{cosce}^2\alpha}} = \frac{3W\sin\alpha}{\sqrt{1+3\sin^2\alpha}}\ .$$

Ex. 31. There uniform rods AB, BC and CD of weight W, $2W$ and W respectively are freely jointed at B and C and are freely hinged to fixed points at A and D. They rest in equilibrium with AD uppermost and horizontal. If $AB=2a$, $BC=4a$ and $AD=6a$, calculate the magnitudes and directions of the reactions at A and B.

[London]

In figure 120, the three rods are shown to be separated from one another to show the actions and reactions at the joints clearly. Let X and Y be the horizontal and vertical components of the reaction at the end B of the rod AB in the directions as shown in the figure. The same forces in the opposite directions are the components of the reaction at the end B of the rod BC. From symmetry, the vertically upward forces acting on BC are

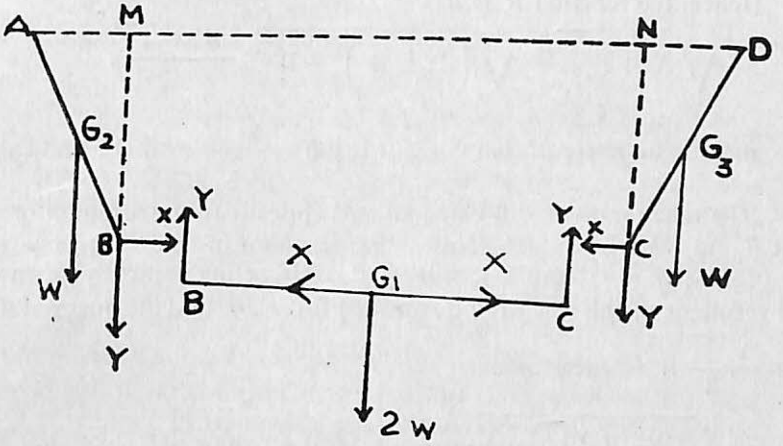


Fig. 120

Y, Y at B and C and the vertically downward force is $2W$ acting at G_1, the middle point of the rod BC. For equilibrium of the rod BC, we have

$$2Y=2W \Rightarrow Y=W.$$

Considering the equilibrium of the rod AB which is acted upon by the forces X, Y at B. W at G_2, the middle point of AB and the reaction of the hinge at A (not shown in the figure) and taking moments of the forces about A, we get

$$X.\ AB \sin\angle BAM = Y.\ AB \cos\angle BAM + W.\ AG \cos\angle BAM,$$

where BM and CN are perpendiculars to the horizontal line AD.

Now, $AM=\frac{1}{2}(AD-BC)=\frac{1}{2}(6a-4a)=a$

$$\therefore\ \cos\angle BAM=\frac{AM}{AB}=\frac{a}{2a}=\frac{1}{2}\ \therefore\ \sin\angle BAM=\sqrt{1-\frac{1}{4}}$$
$$=\frac{\sqrt{3}}{2}$$

Hence, from (1),

$$X.\ 2a.\frac{\sqrt{3}}{2}=Y.\ 2a.\frac{1}{2}+Wa.\ \frac{1}{2}=aW+\frac{1}{2}aW=\frac{3aW}{2}$$

$$\Rightarrow\ X=\frac{\sqrt{3}}{2}W\ .$$

Hence, the reaction R at B

$$=\sqrt{X^2+Y^2}==\sqrt{\left(\frac{\sqrt{3}}{2}W\right)^2+W^2}=\frac{W\sqrt{7}}{2};$$

making an angle of $\tan^{-1}\frac{X}{Y}$ i.e. $\tan^{-1}\frac{\sqrt{3}}{2}$ with the vertical.

The reaction at A will be equal and opposite to the resultant of X and Y at B and W at G_2 of AB. Now, the resultant of the vertically downward forces W at G_2 and Y i. e. W at B is $2W$ acting vertically downwards. The resultant of the vertically downward force $2W$ and the horizontal force $X=\frac{\sqrt{3}}{2}W$ is, therefore,

$$\sqrt{4W^2+\left(\frac{\sqrt{3}}{2}W\right)^2}=\frac{W}{2}\sqrt{19}$$ at an angle of

$$\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}W}{2W}\right)=\tan^{-1}\frac{\sqrt{3}}{4}$$

with the vertical.

EXERCISE 7.1

A. Reduction of coplanar forces

1. Forces 3, 2, 4, and 5 N act respectively along the sides AB, BC, CD, and DA of a square. Find the magnitude of their resultant and the points where its line of action meets AB and AD.
   [Ans. $\sqrt{10}\,N$: If E and F are the required points on BA produced and AD produced, then $AE=2AB$ and $AF=6\,AB$.]

2. $ABCDEF$ is a regular hexagon. Forces P, $2P$, $3P$, $5P$ and $6P$ act along AB, BC, DC, EF and AF respectively. Show that a force can be determined to act along ED so that the six forces are equivalent to a couple and find the magnitude of the moment of the couple.
   $\left[\text{Ans. } \frac{3\sqrt{3}}{2}Pa\text{, where a is the side of the hexagon}\right]$

3. Find the magnitude, direction and line of action of the resultant of the forces 1, 2 and 3 units acting in order round the three sides BC, CA and AB respectively of an equilateral triangle ABC of sides 2a.
   [Ans. $\sqrt{3}$ units acting upwards perpendicular to AC and cutting CB

produced and CA produced at distances 4a from B and a from A respectively.]

4. A, B, C are three points on a line $ABCD$ such that $AB=BC=a$. Forces 3, 6 and $4N$ respectively act at A, B and C in directions making angles of 60°, 120° and 270° with AD. Show that they reduce to a single force and find where its line of action cuts AD.

$\left[\text{Ans. } \dfrac{2(48\sqrt{3}-17)a}{179} \text{ from } A \text{ on } BA \text{ produced.}\right]$

5. AD is an altitude of $\triangle ABC$ in which $BC=6$m, $CA=7$m and $AB=5$m. A force of $12N$ along DA is equilibriated by parallel forces at B and C. If the directions of all the forces are rotated about A, B, C respectively through the same angle so as to be perpendicular to AB, prove that the resultant of the forces is a couple and find its moment.

[Ans. 57.6 Nm.]

6. $ABCD$ is a trapezium in which the parallel sides AD and BC are in the ratio 2:3 and $AB=AD=DC$. Find the magnitude and position of the resultant of forces $3P$ from B towards C, P from B towards A, $2P$ from D towards A and $2\frac{1}{2}$ P from D towards C.

$\left[\text{Ans. } 2.37P, \text{ it cuts } BC \text{ at } \frac{7}{9}\ BC \text{ from } B, \text{ and makes an angle } \tan^{-1}\frac{\sqrt{15}}{5} \text{ with } BC\right]$.

7. $ABCD$ is a rectangular board in which $AB=DC=3$m, $BC=AD=6$m. One rope pulls it along AD with a force of $15N$, another rope pulls it along BC with a force of 25 N, a third rope pulls it along CD with a force of $16N$. Find what force along AB will make the resultant pass through G, the point of intersection of the two diagonals of the rectangle and calculate the resultant completely.

[Ans. A force of 36 N from B to A; 53 N nearly.]

8. A unit force acts along the side, AB, of a square $ABCD$. Find the magnitudes and directions of the forces which must act along the remaining three sides in order to maintain equilibrium. Find the resultant if (i) the force along BC is reversed, (ii) the forces along BC and AD are both reversed.

[Ans. 1 from C to D, 1 from C to B, 1 from A to D (i) 2 along BC (ii) a couple of moment twice the side of the square.]

9. $ABCD$ is a square. Four forces whose algebraic magnitudes form an arithmetical progression, act along the sides, taken in order.

Show that, if their resultant passes through a corner of the square, the progression is a diminishing one, in which, if the common difference is $2P$, the greatest force is $5P$ or $3P$.

10. $ABCD$ is a square of side 2a. P is the middle point of AD, and Q that of DC and the following forces act on the square: 20 from A to B, 20 from C to D, 40 from A to Q and 30 from P to B. Show that the resultant is a force of 50 and that the length of the perpendicular from A on its line of action is about 27a.

11. $ABCD$ is a rectangle in which $BC=\mathrm{n}\,AB$. Prove that the system of forces P, nP, P, nP acting along AB, CB, CD and AD would maintain equilibrium. Find, when $BC=3AB$, the magnitude and position of the resultant of forces 1, 3, 1, $5N$ acting along the sides AB, BC, CD and DA respectively of the rectangle.
[Ans. $2N$ parallel to DA cutting BA produced at a distance $3AB$ from A.]

12. Four forces P, P, Q, Q act along the sides AB, BC, CD, DA of a rhombus $ABCD$. Find the sum of their moments about the centre O of the rhombus. Prove that their resultant is at a distance $\frac{1}{2}BD.\dfrac{P+Q}{P-Q}$ from O. Discuss the case when $P=Q$.

[Ans. When $P=Q$, they are equivalent to a couple.]

13. O is any point inside a triangle ABC. α, β, γ are the perpendiculars from O on the sides. If the forces acting along the sides BC, CA and AB in the same sense be equal to $\dfrac{K}{\alpha}$, $\dfrac{K}{\beta}$, $\dfrac{K}{\gamma}$ respectively, show that the equation of the line of action of the resultant force is

$$x\left(\frac{\sin B}{\gamma}-\frac{\sin C}{\beta}\right)+y\left(\frac{1}{\alpha}-\frac{\cos C}{\beta}+\frac{\cos\beta}{\gamma}\right)+3=0$$

with reference to O as the origin, a line through O parallel to BC as the-x-axis and a line through O perpendicular to BC as the y-axis.

14. Prove that a force acting in the plane of the triangle ABC can be replaced uniquely by three forces acting along the sides of the triangle.

15. Show that a system of coplanar forces can be reduced to
   (i) two forces acting through two given points
   (ii) two forces, one of which acts through a given point, and the other along a given straight line.

16. If six forces of magnitudes 1, 2, 3. 4, 5, and $6N$ act along the sides of a regular hexagon, taken in order, show that the single equivalent

force is of magnitude $6N$ and that it acts along a line parallel to the force $5N$ at a distance from the centre of the hexagon $3\frac{1}{2}$ times the distance of the side from the centre.

[Patna 1957]

17. The sum of the moments of a system of coplanar forces about each of three non-collinear points in the plane of the forces is a non-zero constant. Prove that the system is equivalent to a couple.

Hence show that three forces represented in magnitude, direction and position by the three sides of a triangle, taken in order, are equivalent to a couple.

[C.U. 1937].

18. The algebraic sum of the moments of a system of coplanar forces, not in equilibrium, is zero about each of two points A and B. Show that the algebraic sum of the resolved parts of the system of forces in the direction perpendicular to AB is zero.

19. The algebraic sum of the moments of a system of coplanar forces about the points whose co-ordinaies are (1, 0) (0, 2) and (2, 3) referred to rectangular axes are G_1, G_2 and G_3 respectively. Find the tangent of the angle which the direction of the resultant force makes with the axis of x.

[Lucknow, 1939; C.H. 1956]

$$\left[\text{ Ans. } \frac{G_1-3G_2+2G_3}{2G_1-G_2-G_3}\right]$$

20. The moments of a system of coplanar forces, not in equilibrium, about three collinear points A, B and C in their plane are G_1, G_2, G_3 respectively. Prove that (with due regard to the sign)

$G_1.BC+G_2.\ CA+G_3.\ AB=O.$ [P.U. 1939; Punjab 1939; Delhi 1962]

21. The moments of a force acting in the plane of the rectangular axes Ox and Oy about the points (0, 0), (10, 0) and (0, 5) are 368,—92 and 438 units respectively. Determine the components of the force parallel to Ox and Oy and also the equation of its line of action.

[C. U. 1962].

[Ans. 14 and 46; 23x+7y=184].

22. The moments of a force lying in the plane of the triangle ABC about A, B and C are L, M and N respectively. If the force is the resultant of three forces P, Q and R acting in the same sense along BC, CA and AB respectively, then prove that

$P:Q:R=aL:bM:cN.$

23. *ABC* is an equilateral triangle and *D*, *E*, *F* are the mid-points of the sides *BC*, *CA*, *AB* respectively. Forces *P*, 2*P* and 3*P* act along *BC*, *CA* and *AB* and forces 4*P*, 5*P* and 6*P* act along *FE*, *ED* and *DF*. Find the line of action of the resultant.

[Ans. Parallel to *CB* dividing *DA* in the ratio 1:5]

24. *ABCD* is a quadrilateral in which the sides *BC* and *AD* are parallel. If the forces p.*AB*, q.*BC*, r.*CD* and s.*DA* acting along *AB*, *BC*, *CD* and *DA* respectively are equivalent to a couple, show that p=r and (p−s) *AD*=(r−q) *BC*.

25. A system of forces acts on an equilateral triangular plate *ABC* of side 2a. The forces act in the plane of the plate and their moments about *A*, *B* and *C* are G_1, G_2 and G_3 respectively. Show that the magnitude of the resultant is

$$\left[\frac{G_1^2+G_2^2+G_3^2-G_1G_2-G_2G_3-G_3G_1}{3a^2}\right]^{\frac{1}{2}}.$$

26. A system of coplanar forces reduces to a single force and a couple of moment *G* referred to the origin. Referred to the points (x_1, y_1) and (x_2, y_2), the couples have momenns G_1 and G_2 respectively. Show that the equation of the line of action of the resultant is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x & y & 0 \\ x_1 & y_1 & G_1 \\ x_2 & y_2 & G_2 \end{vmatrix}$$

27. The algebraic sum of the moments of a system of coplanar forces about three non-collinear points *A*, *B*, *C* in their plane are *L*, *M*, *N* respectively. Prove that their resultant *R* is given by

$$R^2=\frac{\Sigma a^2(L-M)(L-N)}{4\triangle^2},$$

where a, b, c are the sides of the triangle and $\triangle$, its area.

28. Six coplanar forces act on a body along the sides *AB*, *BC*, *CD*, *DE*, *EF* and *FA* of a regular hexagon *ABCDEF*, in which *AB* is 1m long. Their magnitudes are 10, 20, 30, 40, *P* and *Q N* respectively. Find *P* and *Q* so that the system reduces to a couple.

[P.U. 1930]

[Ans. $P=-10N$, $Q=60\ N$]

29. A force has moments 6 units, −26 units and 36 units about the origin, the points (8,0) and (0, 10) respectively. Find the magnitude and the line of action of the force.

[Ans. 5 units of force in a direction joining the points (0,−2) and $(\frac{3}{2}, 0)$].

30. The moments of a system of forces acting in the x, y plane about (0,0), (a, 0) and (0, a) are aw, 2aw and 3aw respectively. Find the components parallel to the co-ordinate axes and the line of action of the single force to which the system is equivalent.

[C.U. 1965; C.U. 1960]

[Ans. $2W, -W, x+2y+a=0$]

31. The straight line $4x+3y=5$ meets the rectangular axes Ox and Oy at the points A and B respectively. If the forces P, Q and R act along the lines OB, OA and AB respectively, find the magnitude of the resultant and the equation of its line of action. [C.H. 1958]

[Ans. $\sqrt{P^2+Q^2+R^2+\frac{8}{5}PR-\frac{6}{5}QR}$;
$x(5P+4R)-y(5Q-3R)-5R=0$]

32. Forces P, Q and R acting along the altitudes from A, B and C of the triangle ABC are equivalent to forces X, Y and Z acting along BC, CA and AB respectively.

Prove that

$$\frac{X}{\cos A}=\frac{Q}{\sin B}-\frac{R}{\sin C}$$

and two similar equations.

[Allahabad 1923]

33. $ABCD$ is a square whose side is 2 units in length. Forces a, b, c, d act along the sides AB, BC, CD and DA, taken in order, and forces $p\sqrt{2}, q\sqrt{2}$ act along AC and BD respectively. Show that if $p+q=c-a$, and $p-q=d-b$, the forces are equivalent to a couple of moment $a+b+c+d$. [Madras 1940]

34. If a system of forces in one plane reduces to a couple of moment G and when each force is turned round its point of application through a right angle, it reduces to a couple H. Prove that when each force is turned through an angle θ, the system is equivalent to a couple whose moment is $G\cos\theta+H\sin\theta$.

35. Forces $l.\ BC$, m. CA and n. AB, where l, m, n are positive, act along the sides BC, CA and AB respectively of a triangle ABC, in the directions indicated by the order of the letters. Show that the line of action of their resultant divides BC, CA and AB externally in the ratio m: n, n:l and l:m respectively.
What happens if $l=m=n$?

36. Find the intercepts made on the rectangular axes OX and OY by the line of action of the resultant of a force of 7 units along OP,

where the co-ordinates of P are $(3, 4)$, and a counter-clockwise couple of moment 21 units.

$$\left[\text{Ans. } 3\frac{3}{4},\ -5\right]$$

37. Three forces, each equal to P, act along the sides of a triangle ABC, taken in order. Prove that the resultant R is given by

$R=P(1-8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C)^{\frac{1}{2}}$ and find the distance of its line of action from A. Examine the case when the triangle is equilateral.

$$\left[\text{Ans. } \frac{2P}{aR}\,\triangle,\ \text{where } \triangle \text{ is the area of } \triangle\, ABC\right]$$

B. Three forces in equilibrium

38. A uniform rod can turn freely about a hinge at one of its ends, and is pulled aside from the vertical by a horizonal force acting at the other end of the rod and equal to three quarters the weight of the rod. Find the inclination of the rod to the vertical and also the reaction at the hinge.

$\left[\text{Ans. } \tan^{-1}\left(\frac{3}{2}\right);\ 1\frac{1}{4}\right.$ times the weight of the rod at $\tan^{-1}\left(\frac{4}{3}\right)$ to the horizontal $\Big]$.

39. A heavy uniform rod is suspended from a point O by two strings OA and OB. Show that the plane OAB is vertical. [C. U. 1925].

40. A smooth sphere rests against a smooth vertical wall being supported by a string attached to it and to a point vertically above the point of contact. If the length of the string is equal to the radius of the sphere, find the tension of the string and the pressure on the wall.

$\left[\text{Ans. Tension} = \frac{2W}{\sqrt{3}}\right.$ and pressure$=\frac{W}{\sqrt{3}}$, where W is the weight of the sphere $\Big]$.

41. A uniform rod of length a rests with one end against a smooth vertical wall, the other and being tied to a point of the wall by a string of length l. Prove that the rod may remain in equilibrium at an angle θ to the wall, given by

$$\cos^2\theta = \frac{l^2-a^2}{3a^2}\,.$$

[C.U. 1941.]

42. A uniform beam, AB, 6m long, weighs $40N$. The end A, about which the beam can turn freely, is attached to a vertical wall, and the beam is kept in a horizontal position by a rope attached to a point of the beam $1\frac{1}{4}$ m from A and to a point of the wall vertically above A. If the tension of the rope is not to exceed 120 N, show that the height above A of the point of attachment of the string to the wall must not be less than $1\frac{2}{3}$ m.

43. AB is a uniform bar of weight W, movable about a smooth horizontal axis fixed at A; to B is attached a light cord which passes over a pulley C fixed vertically over A, and supports a weight P at its free end. Show that, in the position of equilibrium, $CB=\frac{2P}{W}\cdot AC$.

44. Show that it is impossible for a heavy rod to rest in equilibrium with its ends on two smooth planes, one of which is horizontal and the other inclined to the horizontal at any angle.

45. A uniform rod has its lower end fixed to a hinge, and its other end attached to a string which is tied to a point vertically above the hinge. Show that the direction of the action at the hinge bisects the string.

46. A rod, AB, hinged at A, is supported in a horizontal position by a string BC, attached at the end B, so that C is a fixed point in the vertical through A. If $AC=AB$ and the weight of the rod is 20 N, find the reaction of the hinge.

[Ans. $10\sqrt{2}\,N$ above AB making an angle of 45° with it.]

47. Prove that a uniform rod cannot rest entirely within a smooth spherical bowl, except in a horizontal position.

48. A uniform heavy rod, AB, freely hinged to a fixed point A in a smooth wall is kept in a horizontal position parallel to the wall by a light cord, attached to its free end B and fastened to a point P in the line LM in which the wall meets the ceiling. Prove that for various positions of P on LM (the length of the cord being adjusted so that the rod is always horizontal), the tension in the cord is proportional to its length BP.

49. A uniform rod, AB, is suspended with its end in contact with a smooth vertical wall AC by a string CE. If $AE=\frac{1}{3}AB$, show that CB will behorizontal.

[P.U. 1928].

50. *A* rod, *AB*, hinged at an extremity *A*, is acted upon by a horizontal force which is $\frac{2}{\sqrt{3}}$ times the weight of the rod, acting at a point *C* of the rod such that in the position of equilibrium its inclination to the vertical is 60°. Find the position of *C*.

[Ans. $BC=\frac{1}{4}\,AB$].

51. A uniform rod, 3 m long, is suspended by a light string of length 5 m passing over a smooth peg and rests horizontally, the string being attached to the ends of the rod. If the rod weighs 7*N*, find the tension of the string.

$\left[\text{ Ans. } 4\frac{3}{8}N \right]$.

52. A uniform rod, *CD*, movable about a smooth hinge *C*, presses against a smooth plane *AB* of inclination α to the horizontal. Find the inclination of the rod to the horizon in order that the pressure exerted by it on the plane may be equal to half its own weight.

[Ans. $\frac{1}{2}\alpha$].

53. A uniform rod of weight *W* and length 2*l* has one end against a smooth vertical wall and rests at an inclination of 45° with the vertical upon a smooth rail parallel to the wall. Find the distance of the rail from the wall, and the reactions.

[C.U. 1914].

[Ans. $\frac{1}{4}\,l\sqrt{2}$; *W* (of the wall) and $W\sqrt{2}$ (of the rail)]

54. A uniform rectangular plane *ABCD* of weight *W* hangs from *A*. If a weight of *W′* be suspended from *B*, show that the inclination θ of the diagonal *AC* to the vertical is given by

$$\tan\theta=\frac{2\,\text{ab}\,W'}{W(\text{a}^2+\text{b}^2)+2W'\,\text{a}^2},$$

where *AB*=a and *AD*=b.

[The weight of a rectangular plane acts through the point of intersection of its two diagonals].

55. *A* uniform rod, *AB*, is inclined at an angle of 60° to the vertical with one end *A* resting against a smooth vertical wall, being supported by a string attached to a point *C* of the rod distant 2m from *B* and also to a ring in the wall vertically above A. If the length of the rod be 8m, find the position of the ring and inclination and tension of the string.

$\Big[$ Ans. Tension $=\frac{2\,W}{\sqrt{3}}$, where *W* is the weight of the rod.

The ring is at a height of 6 m from A and the inclination of the string with the downward vertical is 30°.]

56. A heavy uniform rod, 15 m long, is suspended from a fixed point by strings fastened to its ends, their lengths being 9m and 12m. If θ be the angle at which the rod is inclined to the vertical, show that $25 \sin \theta = 24$.

[Punjab 1933].

57. A heavy uniform rod of length 2a rests in equilibrium, having one end against a smooth vertical wall. and being placed upon a peg at a distance b from the wall. Show that the inclination of the rod to the horizon is

$$\cos^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

58. *A* uniform rod, *AB*, of weight W can turn freely about a hinge at A and to the end B is attached a string which passes over a small smooth pulley at C, vertically above A, and carries a weight w hanging freely. Prove that in the position of equilibrium, $BC:AC=2w:w$.

59. A uniform beam of weight W can move freely in a vertical plane about a hinge at one end A. To the other end B, a string is fastened which passes aver a small fixed smooth pulley vertically above A, and supports a weight w at the other end. Prove that the beam can rest inclined to the horizontal at an angle whose sine is

$$\frac{l}{2a}\left(1-\frac{4\,w^2}{W^2}\right)+\frac{a}{2l},$$

where a is the length of the beam and l is the height of the pulley above A.

[Lucknow 1931].

60. A piece of wire of weight 4 units is bent to form the sides AC, CB of an equilateral triangle. If a weight of 1 unit is attached to B and the wire is suspended from A, show that, in the position of equilibrium, BC is horizontal.

61. Two strings have each one of their ends fixed to a peg, and the others to the ends of a uniform rod. When the rod is hanging in equilibrium, show that the tension of the strings are proportional to their lengths.

62. A straight rod, AB, whose centre of gravity (i.e. the point in the rod, through which the weight of the rod passes) is 7 cm from A and 5 cm from B is suspended from a fixed point C by means of two strings AC and BC, of lengths 13 cm and 7 cm respectively. Show that the angle that the rod makes with the vertical in the position of equilibrium is 60° and find the tensions of the strings in terms of W, the weight of the rod.

$$\left[\text{Ans. } \frac{65\,W}{96},\ \frac{49\ W}{96}\right].$$

63. A heavy uniform beam is hung from a fixed point by two strings attached to its ends. If the lengths of the strings and the beam be as 2: 5: 6, find the ratio between the tensions of the strings and the weight of the beam.

[Ans. 2: 5: $\sqrt{22}$].

64. A uniform beam of length l and weight W hangs from a fixed point by two strings of lengths a and b. Prove that the inclination of the rod to the horizon is

$$\sin^{-1}\left(\frac{a^2-b^2}{l\sqrt{2(a^2+b^2)-l^2}}\right).$$

Find also the tensions of the string.

$$\left[\text{Ans. } \frac{Wa}{\sqrt{2(a^2+b^2)-l^2}},\ \frac{W\,b}{\sqrt{2(a^2+b^2)-l^2}}\right]$$

65. Equal weights, P and P, are attached to two strings ACP and BCP passing over a smooth peg C. AB is a heavy beam, of weight W, whose centre of gravity (i.e. the point in the beam, through which the weight of the beam acts) is at distances of a from A and b from B. Show that AB is inclined to the horizon at an angle

$$\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

66. A uniform bar, AB, of weight $2W$ and length l, is free to turn about a smooth hinge at its upper end A, and a horizontal force is applied to the end B so that the bar is in equilibrium with B at a distance a from the vertical through A. Prove that the reaction at the hinge is

$$W\left(\frac{4l^2-3a^2}{l^2-a^2}\right)^{\frac{1}{2}}.$$

67. A uniform rod, AB, can turn freely in a vertical plane about the end A, which is fixed, and the rod is held at an inclination θ to the vertical by means of a string attached to B. Find the direction of

the string so that the tension in it may be as small as possible, and determine the reaction of the hinge in this case.

[Ans. Perpendicular to AB; $\frac{W}{2}\sqrt{4-3\sin^2\theta}$, where W is the weight of the rod.]

68. A uniform rod, AB, is in equilibrium at an angle α with the horizontal with its upper end A resting against a smooth peg and its lower end B attached to a light cord which is fastened to a point C on the same level as A. Prove that the angle β at which the cord is inclined to the horizontal is given by the equation $\tan\beta = 2\tan\alpha + \cot\alpha$ and that $AC = \frac{AB\sec\alpha}{1+2\tan^2\alpha}$.

[Delhi 1940]

69. A uniform rod whose weight is W is supported by two fine strings, one attached to each end, which after passing over small fixed smooth pulleys, carry weights W_1 and W_2 respectively at the other ends. Show that the rod is inclined to the horizon at an angle

$$\sin^{-1}\left(\frac{W_1^2-W_2^2}{W\sqrt{2(W_1^2+W_2^2)-W^2}}\right).$$

70. A uniform rod of weight W rests with its ends in contact with two smooth planes inclined at angles α and β respectively to the horizon and intersecting in a horizontal line. If θ be the inclination of the rod to the vertical, show that $2\cot\theta = \cot\beta - \cot\alpha$. Also find the reactions at the ends of the rod.

[P.U. 1933]

$$\left[\text{Ans. } \frac{W\sin\beta}{\sin(\alpha+\beta)},\quad \frac{W\sin\alpha}{\sin(\alpha+\beta)}\right]$$

71. A uniform bar of length a rests suspended by two strings of lengths l and l' fastened to the ends of the bar and to two fixed points in the same horizontal line at a distance b apart. If the directions of the strings produced meet at right angles, and if T_1 and T_2 be the tensions of the strings, prove that

$$\frac{T_1}{T_2} = \frac{\mathrm{a}l+\mathrm{b}l'}{\mathrm{a}l'+\mathrm{b}l}.$$

72. A uniform rod, 10 m long and weighing 40 N, is placed on two smooth planes inclined at 30° and 60° to the horizontal. Find the

pressures on each plane and the inclination of the rod to the horizontal when in equilibrium.

[Ans. $20\sqrt{3}\ N$ on 30° plane and $20\ N$ on 60° plane. Inclination of the rod is 30° to the horizontal.]

73. A uniform rod, AB, of weight W is hinged at the end A, and to the other end B is attached a string which passes over a small pulley at C and carries a weight W' hanging freely. If AC be horizontal and $AC=AB$, show that, when $\angle CAB=60°$, equilibrium will be maintained if $6W'=\sqrt{3}\,W$. Find also the reaction at the hinge.

$\left[\text{Ans. } \sqrt{\frac{7}{12}}W\right.$ in a direction making an angle $\sin^{-1}\sqrt{\frac{7}{14}}$ with the upward drawn vertical. $\left.\right]$

74. A picture of weight 5 units is hung from a nail by a cord of length 5 units to two rings attached with the picture, the two rings being 3 units of length apart. Find the tension in the cord.

$\left[\text{Ans. } 3\frac{1}{8}\text{ units}\right]$

75. A uniform square lamina rests in equilibrium under gravity in a vertical plane with two of its sides in contact with smooth pegs in the same horizontal line at a distance c apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by $c\,(\sin\theta+\cos\theta)=a$, where $2a$ is the length of a side of the square.

[C.U. 1946].

76. A square of side 2a is placed with its plane vertical between two smooth pegs which are in the same horizontal line and at a distance d. Show that it will be in equilibrium when the inclination of one of its edges to the horizon is either $\frac{\pi}{4}$ or $\frac{1}{2}\sin^{-1}\left(\frac{a^2-d^2}{d^2}\right)$.

77. A uniform triangular lamina ABC has a right angle at C and rests in a vertical plane with its sides in contact with two smooth pegs, D and E, in the same horizontal plane, the vertex C being downwards. Prove that the inclination θ of AC to the horizontal is given by $AC\cos\theta-BC\sin\theta=3\,DE\cos 2\theta$.

[Agra 1946, 1951]

78. A heavy equilateral triangle, hung upon a smooth peg by a string, the ends of whichh are attached to two of its angular points, rests

with one of its sides vertical. Show that the length of the string is twice the altitude of the triangle.

79. A uniform circular plate of weight W, whose centre is C and plane vertical, is freely movable in its own plane about a horizontal axis fixed at a point A of the circumference. The line AC is to be kept at a given inclination α to the vertical by causing the plate to rest against a fixed smooth peg at a point B on the circumference. Find the position of B such that the pressure on the peg is least, and find this least pressure.

[Ans. $\angle ACB=90^\circ$ and B is below C. Least pressure$=W \sin \alpha$]

80. A uniform square lamina of side 2a rests in a vertical plane on two smooth pegs in a horizontal line. Show that if the sum of the distances of the pegs from the lowest corner is equal to a, there is equilibrium.

81. A uniform rectangular board rests vertically in equilibrium with its sides a and b on two smooth pegs in the same horizontal line at a distance c apart. Prove that the side of length a makes with the vertical an angle θ given by $2c \cos 2\theta = b \cos \theta - a \sin \theta$.

[Agra 1956; Baroda 1954].

82. $ABCD$ is a uniform lamina in the shape of a rhombus with side of length a and the angle $A=2\alpha$. P and Q are two smooth pegs, PQ being of length l and horizontal. Find the angle that AC makes with the vertical if the lamina can rest with points on the sides AB, AD in contact with the pegs, and with AC, not vertical. Show that such a position of equilibrium occurs only if $a \cos^2 \alpha \sin \alpha < l < a \cos^2\alpha$.

[Gorakhpur 1962]

83. A rod of length l rests wholly inside a fixed smooth hemispherical bowl of radius a placed with its axis vertical. The centre of gravity of the rod divides its length in the ratio m:n. Show that the inclination of the rod to the horizon is

$$\sin^{-1}\left[\frac{(n-m)l}{2\sqrt{(m+n)^2 a^2 - mnl^2}}\right].$$

84. Show that the greatest inclination to the horizon at which a uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl placed with its rim horizontal, is $\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$.

85. A uniform rod rests on a fixed smooth sphere with its lower end pressing against a smooth vertical wall which touches the sphere. If

θ be the angle at which the rod makes with the vertical, when in equilibrium, prove that $a=2l \sin(\frac{1}{2}\theta) \cos^3(\frac{1}{2}\theta)$, where l is the length of the rod and a, the radius of the sphere.

[Lucknow 1956].

86. A smooth hemispherical bowl of radius r is placed on the ground with its rim in contact with a smooth vertical wall. A heavy uniform rod is placed with one end inside the bowl, and the other in contact with the wall. If θ be the inclination to the horizon at which the rod rests, prove that the length of the rod is

$$r \sec\theta\left(1+\frac{1}{\sqrt{1+4\tan^2\theta}}\right).$$

87. A hemispherical bowl of radius r rests on a smooth table and partly inside it rests a rod of length $2l$ and of weight equal to that of the bowl. Show that the position of equilibrium is given by the equations

$$l \sin(\alpha+\beta)=r \sin\alpha=-2r\cos(\alpha+2\beta),$$

where α is the inclination of the base of the hemisphere to the horizon and 2β is the angle subtended at the centre by the part of the rod within the bowl.

88. A smooth bowl in the form of a part of a sphere is placed with its axis vertical, and a rod rests with one end within it, a part of it projecting out over the rim. If α be the angle made by any radius to the rim with the vertical axis and β, that made with the same axis by the radius to the lower extremity of the rod in the position of equilibrium, prove that the length of the rod is $4a \sin\beta \sec\frac{1}{2}(\alpha-\beta)$.

[Punjab 1932].

89. A heavy uniform rod of length $2l$ is resting in contact with a smooth sphere of radius r, and the lower end of the rod presses against a smooth vertical wall. If the distance of the centre of the sphere from the wall is c and the rod makes an angle α with the wall, prove that $l\sin^3\alpha+r\cos\alpha=c$.

90. A heavy uniform sphere rests on two smooth inclinded planes whose inclinations to the horizontal are α and β. If α is given, and the pressure on this plane is half the weight of the sphere, prove that β must be equal to $\tan^{-1}\left(\frac{\sin\alpha}{2-\cos\alpha}\right)$.

91. A solid cube, with edges of length l has a string attached to the middle of one edge, the other end of the string, which has also a length l, being fastened to a point in a smooth vertical wall. Find the inclination of the string to the wall when the cube hangs freely.

$$\left[\text{Ans. } \tan^{-1}\frac{1}{3}\right]$$

92. A cylinder of radius r whose axis is fixed horizontally touches a vertical wall along a generating line. A flat beam of uniform material of length $2l$ and weight W, rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. Show that in absence of friction, $\frac{l}{r}=\frac{\sqrt{5}-1}{\sqrt{10}}$, that the pressure on the wall is $\frac{1}{2}W$ and that the reaction of the cylinder is $\frac{1}{2}\sqrt{5}\ W$.

93. A smooth hemispherical bowl of diameter a is placed so that its edge touches a smooth vertical wall. A heavy uniform rod in equilibrium is inclined at an angle of 60° to the horizon with one end resting on the inner surface of the bowl and the other end resting against the wall. Show that the length of the rod must be $a+\frac{a}{\sqrt{13}}$

94. In a smooth hemispherical cup is placed a heavy rod, equal in length to the radius of the cup, the centre of gravity of the rod being one-third of its length from one end. Show that the angle made by the rod with the horizontal is $\cot^{-1}(3\sqrt{3})$

95. A right circular cone of height h and vertical angle 2α is placed with its vertex in contact with a smooth vertical wall, and its slant side resting against a smooth horizontal rail fixed parallel to the wall and at a distance c from it. Show that if, in the position of equilibrium, the axis makes an angle θ with the horizon, then $3h=4c\sec^2(\theta-\alpha)\sec\theta$.

[Madras 1935]

96. The altitude of a cone is h and the radius of its base is r. A string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg; if the cone rests in equilibrium with its axis horizontal, show that the length of the string must be $\sqrt{h^2+4r^2}$. [Lucknow 1954].

c. Equilibrium of any Number of Forces

97. A ladder of weight $24N$ rests on a smooth horizontal ground leaning against a smooth vertical wall at an inclination of $\tan^{-1} 2$ with the horizon and is prevented from slipping by a string attached at its lower end and to the junction of the wall and the floor. A child of weight $72N$ begins to ascend the ladder. If the string can bear a tension of $30N$, how far along the ladder can the child rise with safety ?

$\left[\text{Ans. } \frac{2}{3} \text{ of the length of the ladder.}\right]$

98. A heavy uniform rod of weight W is hung from a point by two equal strings, one attached to each end of the rod. A weight w is hung half-way between the centre and one end of the rod. Prove that the ratio of the tensions in the strings is
$\frac{2W+3w}{2W+w}$

99. A uniform beam rests with its ends on two smooth inclined planes which make angles of 30° and 60° with the horizontal respectively. A weight equal to twice that of the beam can slide along its length. Find the position of the sliding weight when the beam rests in a horizontal position.

$\left[\text{Ans. } \frac{1}{8} \text{ of the length of the beam from the end on } 30^\circ \text{ plane.}\right]$

100. A uniform beam whose weight is 200 N and which is 4m long is hinged to a vertical wall. A string attached to the other end keeps the beam horizontal and is fixed to the wall 3 m above it. A weight of 300N is hung from this end. Find the tension of the string and the thrust on the beam.

$\left[\text{Ans. } 666\frac{2}{3}\, N \text{ and } \frac{100}{3}\sqrt{265}\, N\right]$

101. A ladder resting on a smooth floor and against a smooth vertical wall is prevented from slipping by a rope tied to a point on it with its other extremity fixed at the junction of the floor and the wall. If the centre of gravity (i.e the point of the ladder through which its weight acts) divides it in the ratio m:n, the ratio being measured from the lower part of the rod to the upper part and if the ladder and the rope be inclined at angles θ and ϕ respectively to the horizon, show that the tension of the rope is
$\frac{Wm\cos\theta}{(m+n)\sin(\theta-\phi)}$, where W is the weight of the ladder.

102. A smooth uniform ladder rests with its extremities against a vertical wall and a horizontal plane and is held by a rope, one end of which is attached to a rung of the ladder, one quarter of the way up, the other end being fixed to a point of the base of the wall, vertically below the top of the ladder. Show that if the base and the top of the ladder be distant a and b respectively from the base of the wall, the ratio of the reactions P and Q between the ladder and the ground, and between the ladder and the wall respectively is given by $\frac{P}{Q}=\frac{5b}{3a}$.

103. A square lamina $ABCD$ of weight W is hinged to a vertical wall at A with its plane vertical. A weight W is suspended from its corner C and is supported with AC horizontal by means of a horizontal string joining B to the wall. Find the tension of the string and the reaction at the hinge.

$\left[\text{Ans. } 3W;\ W\sqrt{13} \text{ making an angle } \tan^{-1}\frac{2}{3} \text{ with the horizon.}\right]$

104. A gate weighing W is hung on two hinges a metre apart in a vertical line which is b metre from the centre of gravity of the gate. Find the magnitude of the reaction at each hinge on the assumption that the whole weight of the gate is borne by the lower hinge.

[Ans. Reaction at the lower hinge $=\frac{W}{a}\sqrt{a^2+b^2}$ making an angle of $\tan^{-1}\frac{b}{a}$ with the vertical. Reaction at the upper hinge$=\frac{b}{a}W$ which is horizontal.]

105. A gate 2m high and $\frac{8}{3}$ m wide weighs 560 N and is supported by two hinges $\frac{1}{3}$ m from the bottom and top respectively. The lower hinge can only exert a horizontal reaction. Find the magnitudes of the reactions at both the hinges, if a boy of weight 210 N is sitting on the end of the gate.

[Ans. The reaction at the upper hinge$=1356N$ nearly and the reaction at the lower hinge$=1080N$.]

106. Forces P, Q, R and S acting along AB, BC, CD and DA of a quadrilateral $ABCD$ are in equilibrium. Show that

$$\frac{P\times R}{AB\times CD}=\frac{Q\times S}{BC\times DA}.$$

107. A rod is movable in a vertical plane about a hinge at one end and at the other end is fastened a weight equal to half the weight of the rod. This end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is $\frac{lw}{c}$, where w is the weight of the rod.

108. A uniform rod ACB of length 2a and weight W rests against a smooth horizontal plane at A and against a smooth fixed wall at C. It is prevented from slipping by means of a fine light string connecting A with the point D situated in the horizontal plane vertically below C. If AC be of length c and if the rod rests inclined at an angle α to the horizontal, determine the tension of the string and the reaction of the ground and show that A will not remain in contact with the horizontal plane unless $e > a\cos^2\alpha$.

[Ans. Tension$=\frac{Wa \sin 2\alpha}{2c}$.

Reaction at A$=W\left(1-\frac{a}{c}\cos^2\alpha\right)$.]

109. Forces P, Q, R act along the sides BC, CA, AB respectively of $\triangle ABC$ and forces P',Q',R' act along OA,OB,OC respectively, where O is the circum-centre, in the directions indicated by the order of the letters. If the six forces are in equilibrium, show that $P \cos A+Q \cos B+R \cos C=0$

and $\frac{PP'}{a}+\frac{QQ'}{b}+\frac{RR'}{c}=0$. [Rajasthan 1951]

110. A triangle formed by three rods is fixed in a horizontal position and a homogeneous smooth sphere rests on it. Show that the reaction of each rod is proportional to its length.

111. A sphere of weight W and radius r is hung by a string of length l and a uniform rod of weight w and length 2a has one end attached to the same point and can turn freely about it. If the rod rests touching the sphere, show that the angle θ made by the string with the vertical is given by

$$\tan \theta=\frac{Wa \cos^2 \alpha}{Wr+wa \cos \alpha \sin \alpha}.$$

112. Three uniform rods, AB, BC and CD, whose weights are proportional to their lengths a, b, c are jointed at B and C and are resting

in a horizontal position on two smooth pegs, the distance between which is x. Show that

$$x=\frac{a^2}{2a+b}+\frac{c^2}{2c+b}+b.$$

113. Weights W, w and W are attached to points B, C and D respectively of a light string AE, where B, C, D divide the string into 4equal lengths. If the string hangs in the form of 4 consecutive sides of a regular octagon, with the ends A and E, attached to points on the same level, show that $W=(\sqrt{2}+1)$ w. [Annamalai 1936]

114. A weight, W, is attached to an endless string of length l which hangs over two smooth pegs distant c apart in a horizontal line. Prove that the pressure on each peg is

$$W\sqrt{\frac{l-c}{2(l-2c)}}\ .$$

115. AB is a straight rod of length 2a and weight λW with the lower end A on the ground at the foot of a vertical wall AC, B and C being at the same vertical height 2b above A. A heavy ring of weight W is free to move along a string of length $2l$, which joins B and C. If the system is in equilibrium with the ring at the middle point of the string, show that

$$l^2=a^2-b^2.\ \frac{\lambda(\lambda+2)}{(\lambda+1)^2}\ .$$

[Lucknow 1955; Allahabad 1927]

116. A beam of weight W is divided by its centre of gravity G into two portions, AG and GB, whose lengths are a and b respectively. The beam rests in a vertical plane on a smooth floor AC against a smooth wall CB. A string is attached to a hook at C and to the beam at a point D. If T be the tension of the string and α and β be the inclinations of the beam and the string respectively to the horizon, show that

$$T=W.\frac{a\cos\alpha}{(a+b)\sin(\alpha-\beta)}\ .$$ [Punjab 1941]

117. A circular disc, BCD, of radius a and weight w is supported by a band of inappreciable weight and thickness which surrounds the disc along the arc BCD and is fastened at its extremities to the point A in a vertical wall. The portion AD touches the wall, the plane of the disc being at right angles to the wall. If the length of the

band, not in contact with the disc, be $2l$, show that the tension of the band is

$\frac{W}{2} \cdot \frac{a^2+l^2}{l^2}$. [Delhi 1940]

118. A uniform rod of weight W is supported in equilibrium by a string of length l, attached to its ends and passing over a smooth peg. If now a weight W' be suspended from one end of the rod, prove that the system may be placed in equilibrium by sliding a length $\frac{lW}{2(W+W')}$ of the string over the peg.

119. Two uniform beams, AC and BC, of equal lengths, hang in a vertical plane and are inclined to the horizontal at an angle α. Unequal weights, W, W', are pin-jointed at C, and the ends A and B are higed to two fixed points at the same level. Find the reaction at C and prove that its line of action is inclined to the vertical at an angle

$\tan^{-1}\left(\frac{W+W'}{W-W'}\right)\cos\alpha$. [London 1931]

[Ans. $\frac{1}{4}\sqrt{(W+W')^2\cot^2\alpha+(W'-W)^2}$].

120. Two equal heavy rods, each of weight W and length 2a, are freely hinged together and placed symmetrically over a fixed smooth sphere of radius a. Show that the inclination of each rod to the horizontal is given by $r(\tan^3\theta+\tan\theta)=a$.

[Allahabad 1955; C.U. 1964].

121. Two uniform rods, AB and AC, each of length 2a are smoothly jointed at A and rest symmetrically in a vertical plane on two smooth pegs in the same horizontal level at a distance 2c apart. Show that they are in equilibrium if each rod makes with the vertical an angle $\sin^{-1}\left(\frac{c}{a}\right)$.

122. A heavy triangular lamina is suspended with its plane horizontal from a fixed point by strings attached to its corners. Show that the tension of each string is proportional to its length. Prove also that the same result is true even if the plane of the triangle is not horizontal.

123. Two equal uniform rods, AB and AC, each of weight W are freely jointed at A and rest with the extremities, B and C, on the inside of a smooth circular hoop whose radius is greater than the length of

either rod, the whole being in a vertical plane and the middle points of the rods being joined by a light string. Show that if the string is stretched, its tension is $W(\tan\alpha-\tan\beta)$,

where 2α is the angle between the rods, and β, the angle either rod subtends at the centre.

124. A circular lamina is hung up from three points A, B and C on its rim by equal strings attached to a fixed point. If T_1, T_2 and T_3 be the tensions in the strings, prove that

$$\frac{T_1}{\sin 2A}=\frac{T_2}{\sin 2B}=\frac{T_3}{\sin 2C}.$$

125. Two uniform rods, AB and BC, rigidly joined at B so that ABC is a right angle, hang freely in equilibrium from a fixed point A. The lengths of the rods are a and b, and their weights are aw and bw. Prove that if AB makes an angle θ with the vertical, then

$$\tan\theta=\frac{b^2}{a^2+2ab}.$$

[Lucknow 1952]

126. Two equal beams, AB and AC, each of weight W, connected by a hinge at A are placed in a vertical plane with their extremities B and C resting on a horizontal plane. They are kept from falling by strings connecting B and C with the middle points of the opposite sides. Show that the tension of either string is $\frac{W}{8}\sqrt{1+9\cot^2\theta}$,

where θ is the inclination of either beam to the horizon.

[Rajasthan 1948]

Show also that the action of the hinge on either rod is $\frac{3}{4}W\cot\theta$.

127. A sphere rests on three smooth pegs which lie in a horizontal plane and are at distances a, b and c from one another. Prove that the pressures on the pegs are in the ratios

$a^2(b^2+c^2-a^2): b^2(c^2+a^2-b^2): c^2(a^2+b^2-c^2)$.

128. Two light rods, AB and AC, each of length 4m are hinged at A and are placed in a vertical plane with the ends B and C resting on a smooth horizontal plane. B and C are joined by a string 4m long and a weight W is fastened to the rod AC at a point D, h metre above the horizontal plane. Prove that the tension of the string is $\frac{W\lambda}{12}$.

[Banaras 1953].

129. A uniform rod of length 2a and weight W is suspended freely from the end A and a couple of moment $G(<aW)$ in a vertical plane is applied to the rod. Find the angle the rod makes with the vertical and the reaction it exerts on the hinge. [Banaras 1940; Agra 1949]

$$\left[\text{Ans. } \sin^{-1}\left(\frac{G}{aW}\right);\ W\right]$$

130. Two equal uniform ladders, each of length l, freely joined at A, are connected by a horizontal rope PQ and rest on a smooth horizontal plane, a man of weight W climbs a distance x up one of the ladders. If w be the weight of each ladder, 2b=the length of the rope and $AP=AQ=$a, show that the tension of the rope is

$$\frac{Wx+wl}{2a} \cdot \frac{b}{\sqrt{a^2-b^2}}.$$

131. Two equal uniform rods, AB and AC, each of weight W, are smoothly hinged at A and placed in a vertical plane with the ends B and C resting on a smooth horizontal plane. Equilibrium is preserved by a string which attaches C to the middle point of AB. Show that the tension of the string and the reaction of the rods at A are both equal to $\frac{1}{4}W\sqrt{1+9\cot^2\alpha}$ and that each is inclined at an angle $\tan^{-1}(\frac{1}{3}\tan\alpha)$ to the horizon, where α is the inclination of either rod to the horizon.

132. A step ladder in the form of the letter A, with each of its legs inclined at an angle α to the vertical, is placed on a horizontal floor, and is held up by a cord connecting the middle points of its legs, there being no friction anywhere. Show that, when a weight W is placed on one of the steps at a height from the floor equal to $\frac{1}{n}$ of the height of the ladder, the increase in the tension of the cord is $\frac{1}{n}$W $\tan\alpha$.

[Delhi 1950]

133. A light table stands on three equal vertical legs and a weight is placed at the in-centre of the triangle formed by the points of intersection of the legs with the table. Show that the pressures on the legs are proportional to the lengths of the opposite sides of the triangle.

134. Three uniform beams AB, BC and CD of the same thickness and of lengths l, $2l$ and l respectively are connected by smooth hinges at B and C and rest on a perfectly smooth sphere whose radius is $2l$ so

that the middle point of *BC* and the extremities, *A* and *D*, are in contact with the sphere. Show that the pressure at the middle point of *BC* is $\frac{91}{100}$ of the weight of the beam.

135. A uniform circular disc of weight nW has a particle of weight *W* attached to a point on its rim. If the disc be suspended from a point *A* on its rim, *B* is the lowest point. Also if suspended from *B*, *A* is the lowest point. Show that the angle subtended by *AB* at the centre of the disc is

$$2\sin^{-1}\left\{2\left(\text{n}+1\right)\right\}.$$

136. *AB*, *BC* and *CD* are three equal rods jointed at *B* and *C*. The rods *AB* and *CD* rest on two smooth pegs in the same horizontal line so that *BC* is horizontal. If α be the inclination of *AB* and β that of the reaction at *B* to the horizon, prove that $3\tan\alpha\tan\beta=1$.

137. A fine smooth vertical rod is fixed firmly into the top of a fixed smooth sphere of radius a. A bar of length 2b has at one end a ring which slides on the rod and the bar rests on the sphere. Show that, in equilibrium, the angle α which the bar makes with the vertical is given by $a\cos\alpha=b\sin^3\alpha$.

138. One end of a uniform rod is movable about a hinge and the other is supported by a string attached to it. The string is inclined at the same angle θ to the horizontal as the rod. If *W* be the weight of the rod, show that the reaction at the hinge is

$$\frac{W}{4}\sqrt{8+\operatorname{cosec}^2\theta}\,.$$

139. Two uniform rods, each of weight *W* and length a are freely jointed at *A*, and each passes over a smooth peg at the same level. From *A*, a weight *W′* is suspended. Show that in the position of equilibrium, the inclination θ of the rods to the horizon is given by

$$\cos^3\theta=\frac{c(2W+W')}{2Wa}$$, where c is the distance between the pegs.

140. *AOB* is a weightless rod bent to form an angle α at *O*. Weights *P* and *Q* are attached at *A* and *B*, and the whole is suspended from *O*. If $OA=a$, $OB=b$, and if θ be the inclination of *OA* to the

horizontal, prove that

$$\tan\theta = \frac{Pa}{Qb\sin\alpha} + \cot\alpha.$$

141. A solid hemisphere of weight W and radius a is placed with its curved surface on a smooth horizontal table and a string of length $l(l<a)$ is attached to a point on its rim and to a point on the table. Show that the tension of the string is

$$\frac{3}{8}W.\ \frac{a-l}{\sqrt{2al-l^2}}.$$ [Andhra 1954; Lucknow 1953]

142. A uniform sphere of weight W and radius r rests on a smooth inclined plane and is suspended by a string of length l attached to thesphere, the other end of the string being attached to a point in the plane. Show that the string makes an angle $\sin^{-1}\left(\frac{r}{r+l}\right)$ with the plane.

143. A prism whose cross-section is an equilateral triangle rests with its two edges horizontal on smooth planes inclined at angles α and β to the horizon. If θ be the angle that the plane through these edges makes with the verticals, show that

$$\tan\theta = \frac{2\sqrt{3}\sin\alpha\sin\beta+\sin(\alpha+\beta)}{\sqrt{3}\sin(\alpha\sim\beta)}$$

[Delhi 1964; Gorakhpur 1964]

144. Forces act along the four sides of a quadrilateral, equal respectively to a, b, c and d times the lengths of those sides. Show that, if there is equilibrium, ac=bd, and that the further conditions necessary for equilibrium are that the ratios a:b and b:c shall be the ratios in which the diagonals of the quadrilateral are divided at their point of intersection.

[Lucknow 1934]

145. Two uniform spheres of equal weight but unequal radii, a and b, are connected by a cord of length l attached to a point on each surface. They rest in contact, the string hanging over a smooth peg. Show that the two portions of the string make equal angles $\sin^{-1}\left(\frac{a+b}{a+b+l}\right)$ with the vertical.

146. A heavy uniform bar of length 2a and weight w is movable in a vertical plane round a smooth hinge fixed at one extremity. A

heavy smooth sphere of weight W and radius r is attached to the hinge by a cord of length l. The two bodies rest in contact. If θ and ϕ be the inclinations of the cord and the bar to the vertical, show that $\sin(\theta+\phi)=\frac{r}{l+r}$

and $W(l+r)\sin\theta = wa\sin\phi$.

147. Two rods, OA and OB, are fixed in a vertical plane with O uppermost, each rod making an acute angle α with the vertical. Two smooth rings of equal weight w which can slide upon each rod are connected by a light inextensible string, upon which slides a third smooth ring of weight 2w. Show that in the symmetrical position of equilibrium, the ratio of the distance from O of the third ring to the distance from O of either of the others is $\frac{1}{2}(\sec\alpha+\cos\alpha)$.

148. Two spheres weighing $2N$ and $10N$ rest on two smooth planes of inclinations 45° and 30° with the horizontal and press against each other. Prove that the line of their centres is inclined to the horizontal at an angle

$$\tan^{-1}\left(\frac{2\sqrt{3}-1}{3}\right).$$

149. Two smooth spheres of radii a and b of equal density are connected by a light string of length l, the ends of the string being attached to points on the surface of the spheres. The string is slung over a smooth fixed peg and the spheres hang freely in contact with one another. Show that in the position of equilibrium, the peg divides the length of the string in the ratio $b^3(a+l)-a^4 : a^3(a+l)-b^4$.

150. Two smooth uniform spheres of equal weights but of radii 2m and 3m are placed in a fixed hemispherical bowl of radius 8m. Show that in the position of equilibrium, the inclination θ to the vertical of the line joining the centres of the spheres is given by $11\tan\theta=48$.

151. Two smooth spheres each of radius a and weight W lie in contact with a smooth spherical bowl of radius na. Show that the pressure between them is $\frac{W}{\sqrt{n^2-2n}}$.

152. A particle of weight W rests inside a smooth fixed hemispherical bowl whose middle radius is vertical, being supported by a string

which, after passing over the smooth rim of the bowl, hangs vertically carrying a weight P. Show that the angle between the two portions of the string is given by

$$\cos^{-1}\left(\frac{P+\sqrt{P^2+8W^2}}{4W}\right).$$

153. Two smooth balls of the same material, of radii a and b, are placed inside a hemispherical bowl of radius R. Prove that the line joining the centres of the balls will be horizontal if

$$R=\frac{(a+b)(a^2+b^2)}{a^2-ab+b^2}.$$

154. A circular disc of radius r and weight w is placed inside a smooth sphere of radins R $(>r)$ and at a distance d from the centre of the disc is fixed a weight W. Show that in the position of equilibrium, the plane of the disc makes with the horizon an angle whose tangent is

$$\frac{wd}{(W+w)\sqrt{R^2-r^2}}.$$

155. An intextensible string binds tightly together two smooth weightless cylinders of radii R, r. Show that the ratio of the pressure between the cylinders and the pressure by which it is produced is

$$\frac{4\sqrt{Rr}}{R+r}.$$

156. Inside a fixed vertical ring of radius R, there are placed symmetrically two equal small rings of radius r and a third equal ring is placed symmetrically on them. Prove that the rings will remain in contact provided $R<r(1+\sqrt{27})$

[Agra 1957]

157. Two spheres whose radii are a_1 and a_2 rest inside a smooth hollow vertical cylinder of which the external radius is c and the internal radius is b, where $b<a_1+a_2$. Prove that, if the sphere with radius a_1 is the lower, the cylinder will not overturn if its weight exceeds

$$\frac{W}{c}\left(2b-a_1-a_2\right),$$

where W is the weight of the upper sphere.

158. The weight of a gate is W. In one side of the gate, there are fixed two circular headed staples at C and D. In the gatepost, there are two L-shaped staples fixed at A and B in which are fitted the circular headed staples of the gate. Show that the pressure on the

upper staple is either $\frac{W}{\mathrm{b}}\sqrt{\mathrm{a}^2+\mathrm{b}^2}$ or $\frac{W\mathrm{a}}{\mathrm{b}}$.

according as CD is slightly smaller or greater than AB, where the horizontal length of the gate is 2a and $CD=\mathrm{b}$. [Lucknow 1951]

159. Two uniform rods, AB and AC, of weights W_1 and W_2 and of equal length are smoothly hinged at A, and rest with B and C on a smooth horizontal line being kept in equilibrium by an inextensible string joining BD. A weight W is suspended from a point in AC at a distance $\frac{3}{4}$ AC from A. Prove that the tension of the string is $\frac{1}{4}\left(W_1+W_2+\frac{1}{2}W\right)\tan\frac{A}{2}$.

160. A uniform bar of length 2 a and weight W has a cord of length 2b attached to its ends. The cord passes through a small smooth ring C which is fixed against a smooth vertical wall. The bar is placed in a vertical plane perpendicular to the wall with the end A vertically below C. Prove that, in equilibrium, A is at a distance $\sqrt{\mathrm{b}^2-\mathrm{a}^2}$ below C and that the tension of the string is

$$\frac{1}{2}W\left[\frac{2\mathrm{b}}{\sqrt{\mathrm{b}^2-\mathrm{a}^2}}-1\right].$$

[Birmingham 1935]

161. A picture frame rectangular in shape rests against a smooth vertical wall from two points in which it is suspended by parallel strings attached to two points in the upper edge of the back of the frame, the length of each string being equal to the height of the frame. Show that if the centre of gravity of the frame coincides with the centre of the figure, the picture will hang against the wall at an angle $\tan^{-1}\left(\frac{\mathrm{b}}{3\mathrm{a}}\right)$ to the vertical, a being the height and b, the thickness of the picture. [Allahabad 1955]

162. Two equal rods, each of weight w, are freely jointed and their free ends are attached by strings to a fixed point. A circular disc of weight W and radius r rests in the angle between the rods, and the whole hangs in a vertical plane. If 2a be the length of each rod and each string and 2θ, the angle between the rods, prove that

$$\mathrm{r}=2\mathrm{a}\sin^2\theta\tan\theta.\ \frac{3\mathrm{w}+2W}{W}.$$

163. A pole with one end resting on the ground is kept vertical by two ropes attached to fixed points on the ground at equal distances d from the foot of the pole, and in the same plane with it. The ropes are fastened to the pole at heights a and b and the tensions are adjusted so that the vertical reaction on the ground is twice the weight of the pole. Prove that the total reaction on the ground makes an angle θ with the vertical, where

$$\tan\theta = \frac{1}{4}d\left(\frac{1}{a} \sim \frac{1}{b}\right).$$

164. A uniform rod of weight W and length 2a is free to turn about one end A. It is supported by means of a light string of length l, one end of which is attached to a point vertically above and at a distance h from A while the other end is attached to a smooth ring of weight w, through which the rod passes. Show that, in the equilibrium position, the string is inclined at an angle θ to the vertical, where

$$Wal\,(h\cos\theta - l) = w(l^2 - 2hl\cos\theta + h^2)^{3/2}.$$

165. Three equal uniform rods, each of weight W, are smoothly jointed so as to form an equilateral triangle. If the system be supported at the middle point of one of the rods, show that the action at the lowest angle is $\frac{1}{6}\sqrt{3}\,W$ and that at each of others is $W.\sqrt{\frac{13}{12}}$.

166. Three equal uniform rods, each of weight W, are freely jointed at their ends so as to form an equilateral triangle. Find the forces at the joints when the triangle is suspended by a string from the mid-point of one of the sides.

$$\left[\text{Ans. } \frac{\sqrt{13}}{2\sqrt{3}}\,W,\ \frac{\sqrt{13}}{2\sqrt{3}}\,W \text{ and } \frac{W}{2\sqrt{3}}.\right]$$

167. A rhombus framework, $ABCD$, of four uniform smoothly jointed rods, each of length a and weight w is hung up at A and inside it is placed a smooth uniform circular vertical disc of radius r and weight W, resting symmetrically on the two lower rods, BC and DC. If θ be the inclinmation of the rods to the vertical in the equilibrium configuration, prove that $2a\,(W+2w)\sin^3\theta = Wr\cos\theta$.

168. Three smooth pencils of weight W are tied together by a string and laid on a smooth table. The two lower pencils are pressing each

other with a force P. Show that the tension of the string is $P+\dfrac{W}{2\sqrt{3}}$. [Roorkee 1956]

169. Four equal uniform rods are freely jointed to form a rhombus $ABCD$. The rhombus is suspended from the joint, A, and is maintained in the form of a square by means of a rod of negligible weight, joining the middle points of BC and CD. Find the vertical and horizontal components of the reactions at the joints B and D. Find also the thrust along the line joining the middle points of BC and CD.
[Ans. The vertical and horizontal components at each of B and D are W and $\frac{3}{2}W$ respectively and the thrust$=4W$, where W is the weight of each rod.]

170. Two equal heavy beams, AB and AC, each of weight W are smoothly jointed at A, and B is joined by a string to the mid-point of AC. The beams rest with B and C on a smooth horizontal plane. If $\angle BAC=60°$, show that the tension in the string is $\dfrac{W}{2}$.

171. Six equal weightless rods hinged together at their ends form a regular hexagon $ABCDEF$. The rod AB is held horizontally and a weight W is hung from the middle point of DE, the hexagonal shape being maintained by a light rod CF. Show that the stress in the rod CF is $\dfrac{\sqrt{3}}{3}W$.

172. Five weightless rods form a regular pentagon $ABCDE$ and the framework is stiffened by other weightless rods, BD and CE. The system is placed in a vertical plane with CD on a horizontal table and a weight W is hung from A. Prove that the thrust in BD or CE is $W\cot\dfrac{\pi}{5}$.

173. A step ladder of weight $2W$ consists of two equal parts, jointed at the top and held together by a rope half-way between the top and bottom, so that when the rope is tight, the angle between the two halves of the ladder is $2\tan^{-1}\dfrac{6}{13}$. A man of weight $5W$ mounts the ladder and then stops two-thirds of the way up. Neglecting the friction between the ladder and the ground, show that the tension

in the rope is $2W$ and the horizontal and vertical components of the reaction at the hinge are $2W$ and $\frac{5}{3}$W respectively.

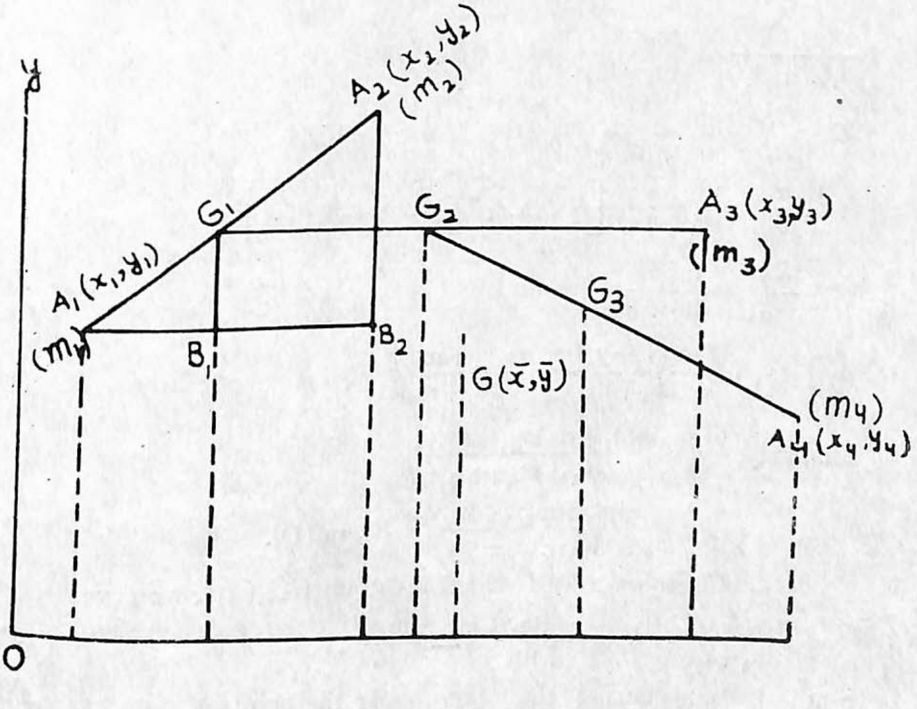
174. A regular pentagon, *ABCDE*, formed by five uniform rods, each of weight *W*, freely hinged to each other at their ends is placed in a vertical plane with *CD* resting on a horizontal plane and the regular pentagonal form is maintained by means of a string joining the middle points of the rods *BC* and *DE*. Prove that the tension in the string is $\left(\cot\frac{\pi}{5}+3\cot\frac{2\pi}{5}\right)$W.

Chapter VIII

Centre of Gravity

8.1 Centre of Mean position or Centroid

Let a system of n scalar magnitudes $m_1, m_2 \ldots m_n$ be associated with a set of n points $A_1, A_2, \ldots A_n$ respectively. Let the co-ordinates of $A_1, A_2, A_3 \ldots A_n$ be (x_1,y_1), $(x_2,y_2), \ldots$ (x_n, y_n) respectively referred to co-ordinate axes Ox and Oy.



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We now divide A_1A_2 at G_1 $(\bar{x}_1, \bar{y}_1)$ such that

$$\frac{A_1G_1}{A_2G_1} = \frac{m_2}{m_1}. \qquad \ldots(A)$$

Similarly, we divide G_1A_3 at $G_2(\bar{x}_2, \bar{y}_2)$ such that

$$\frac{G_1G_2}{A_3G_2} = \frac{m_3}{m_1+m_2}.$$

We again divide G_2A_4 at G_3 $(\bar{x}_3, \bar{y}_3)$ such that

$$\frac{G_2G_3}{A_4G_3} = \frac{m_4}{m_1+m_2+m_3}$$

and so on until we finally arrive at a point G_{n-1} when the last point A_n has been obtainted. We denote this final point G_{n-1} by $G(x,y)$ which is called the *centre of mean position* or *centroid* of the points A_1, $A_2 \ldots A_n$ for the system of n scalar magnitudes m_1, $m_2, \ldots m_n$.

Now, to find the co-ordinates of the centre of mean position, we have from (*A*)

$$\frac{A_1G_1}{A_1G_1+A_2G_1} = \frac{m_2}{m_1+m_2}$$

$$\Rightarrow \frac{A_1G_1}{A_1A_2} = \frac{m_2}{m_1+m_2}$$

$$\Rightarrow \frac{G_1B_1}{A_2B_2} = \frac{m_2}{m_1+m_2} \left[\because \triangle A_1B_1G_1 \sim \triangle A_2A_1B_2 \right]$$

$$\Rightarrow \frac{\bar{y}_1 - y_1}{y_3 - y_1} = \frac{m_2}{m_1+m_2}$$

$$\Rightarrow \bar{y}_1 = y_1 + \frac{m_2(y_2-y_1)}{m_1+m_2} = \frac{m_1y_1+m_2y_2}{m_1+m_2} \qquad (1)$$

Similarly, $\bar{y}_2 = \dfrac{(m_1+m_2)\,\bar{y}_1+m_3y_3}{m_1+m_2+m_3}$

$$= \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3} \quad \text{[from (1)]}.$$

Proceeding in this way till we get the point $G_{n-1}(x,y)$, we have

$$\bar{y} = \frac{m_1y_1+m_2y_2+ \ldots +m_ny_n}{m_1+m_2+\ldots +m_n} = \frac{\Sigma my}{\Sigma m}.$$

Similarly, considering the distances of the points A_1, $A_2 \ldots A_n$ from the y-axis, we have

$$\bar{x} = \frac{m_1x_1+m_2x_2+ \; +m_nx_n}{m_1+m_2+\ldots+m_n} = \frac{\Sigma mx}{\Sigma m}.$$

Thus the co-ordinates of the centre of mean position are given by

$$\left(\frac{\Sigma mx}{\Sigma m}, \frac{\Sigma my}{\Sigma m}\right).$$

Remark : The centre of mean position is a general term which includes, as special cases, the centre of a set of parallel forces and the centre of mass of a body.

8.2 Centre of mass (C.M.)

If we regard the scalar magnitudes $m_1, m_2, \ldots m_n$ in article 8.1 as masses of a system of particles, then the centre of mean position becomes a point called the *centre of mass* (*C.M.*) of the system of particles. However close the particles may be, the proceess of arriving at G, the centre of mass, remains the same. Hence, for a continuous body consisting of minute particles of matter, we can always find its centre of mass. We can thus define the centre of mass of a body as follows:

The centre of mass of a body is the centre of mean position of all the points within it for a system of scalar magnitudes which are proportional to the masses of the indefinitely small particles placed at these points.

The centre of mass of a body is unique so long as there is no relative change of position of the particles constituting the body. A different arrangement of the particles will, however, change the position of the centre of mass of a body.

8.3 Centre of gravity (C.G.)

We know that each particle of a rigid body is attracted to the centre of the earth with a force known as gravity which is proportional to the mass of the particle. We call this force as the weight of the particle. If a rigid body be small compared to the earth, the lines joining the centre of the earth with the different particles of the body will be almost parallel. Hence, the forces acting on different particles of a rigid body can be treated as a system of parallel forces. These parallel forces can be combined into a single resultant force which is called the total weight of the body. This resultant will pass through a fixed point in the body and this fixed point is called the *centre of gravity* of the body which is the same as the centre of the parallel forces. If we turn round the body in any way, the forces of attraction on the different particles will still be parallel, but only their directions relative to the body will change. But the centre of parallel forces acting on different particles is independent of the directions of these forces and depends only on the position of the particles which are fixed relative to the body. Thus the centre of parallel forces formed by the weights of indefinitely large number of indefinitely small particles constituting a body and hence the centre of gravity of the body remains the same in whatever position the body is placed provided that the body is small in comparison with the dimension of the earth.

Thus the centre of gravity of a rigid body may be defined as follows:

The centre of gravity of a body (or a system of particles rigidly connected to one another) is the point fixed in the body (or with respect to the system of particles) through which the resultant weight of the body or the system always passes for all positions of the body.

We observe that the centre of gravity of a very large body whose dimension is not negligible in comparison with the size of the earth (e.g. a big mountain) does not necessarily exist. In this case the lines drawn from different points of the body to the centre of the earth can not be treated as parallel and hence the resultant of all these forces may not necessarily pass through the same point in the body for all positions of the body. Thus the centre of gravity which is taken as a fixed point in the body does not exist. For most of the practical cases, we deal with bodies whose dimensions are small in comparison with that of the earth and so whenever we talk of a rigid body, we shall assume that its centre of gravity exists.

8.4 Uniqueness of the centre of gravity

There can not be more than one C.G. of a body. For, if possible, let a body have two C.G.'s viz. G_1 and G_2. Hence, the weight of the body passes through both G_1 and G_2, in whatever manner the body may be held. We now turn the body such that G_1G_2 is horizontal. The weight which is a vertical force can not now pass through both G_1 and G_2 unless G_1 coincides with G_2. Hence, the body cannot have two G.G.'s. In other words, the C.G. of a body is a unique point in the body.

8.5 Distinction between the centre of mass and the centre of gravity

The weight of a body is due to the action of the force of gravity. But the mass of a body is independent of any attraction to which the body may be subjected. If we imagine that the body has been removed into space where there is no attraction, the body will have no weight and hence the existence of the centre of gravity of the body is beyond our imagination. But the mass of the body will always exist, since it is an inherent property of the material of the body and so the centre of mass of the body even in space devoid of any attraction will exist.

For a body of any shape, the centre of mass always coincides with the centre of gravity whenever such a point exists in the body. Thus a body having its centre of gravity has its centre of mass, but one having its centre of mass may or may not have its centre of gravity.

8.6 Working formulas for finding the centre of gravity of a body or a system of particles

Since the weight of a particle is a force which is proportional to its mass, the farmulas for finding the C.G. of a system of particles will be identical with those for C.M.

Thus if a system of particles of weights $w_1, w_2, w_3 \ldots$ are situated at the points (x_1, y_1), (x_2, y_2), (x_3, y_3),...... respectively referred to the co-ordinate axes Ox and Oy, then the co-ordinates $(\bar{x}, \bar{y})$ of its C.G. are given by the formulas

$$\bar{x} = \frac{\Sigma wx}{\Sigma w}, \quad \bar{y} = \frac{\Sigma wy}{\Sigma w}.$$

Since weights are proportional to masses, these formulas can also be written as

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \bar{y} = \frac{\Sigma my}{\Sigma m},$$ where m denotes the mass.

If, instead of the weights being situated at isolated points as above, we have a continuous distribution of matter, the summations can be replaced by integration. Thus we subdivide the continuous distribution of matter of the body whose centre of gravity is to be determined, into a large number of small parts of such shape that the position ef *C.G.* of each part is known to a first approximation. Then $(\bar{x}, \bar{y})$, the co-ordinates of the *C.G.* of the body, will be approximately given by the above formulas and the larger the number of subdivisions, the more accurate will, in general, be the result. If, therefore, the number of subdivisions be increased indefinitely, the magnitudes of the parts will decrease indefinitely and the sums $\Sigma\, wx$, Σw and $\Sigma\, wy$ will tend respectively to definite integrals

$$\int x dw, \int dw \text{ and } \int y dw$$

(with suitable limits of integration so as to include the whole matter of the body) where δw is the elementary infinitesimal part whose centre of gravity lies at (x,y). Under such circumstances, the above formulas become exact. Hence, we obtain the following formulas for the determination of the centre of gravity of a continuous distribution of matter or a body:

$$\bar{x}=\frac{\int x\,dw}{\int dw},\ \bar{y}=\frac{\int y\,dw}{\int dw},$$

where the limits of integration are to be chosen to include the whole matter or body.

Since $\delta w = g\delta m$, where g is the acceleration due to gravity and m is the mass of the elementary portion of the continuous distribution, the above formulas can be re-written as

$$\bar{x}=\frac{\int x\,dm}{\int dm},\ \bar{y}=\frac{\int y\,dm}{\int dm}.$$

Remark 1 : The formulas

$$\bar{x}=\frac{\Sigma\, wx}{\Sigma w},\ y=\frac{\Sigma\, wy}{\Sigma w}$$

for the *C.G.* of a system of particles can also be derived directly by equating the algebraic sum of the moments of the forces w_1, w_2 $w_3 \ldots$ about x and y axes, taken separately, to the moment of the resultant force $w_1+w_2+\ldots\ldots$ about the same lines respectively.

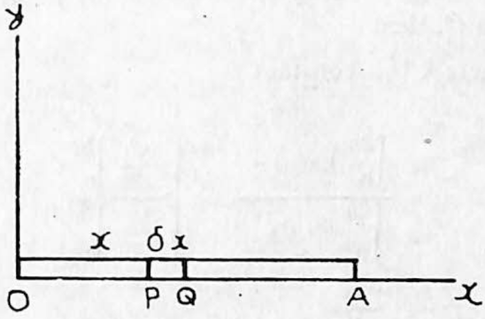
Remark 2 : The positions of the centres of gravity of many bodies may be found easily by observing the points of symmetry of the bodies. Thus the centre of gravity of two equal particles is at the middle point of the line segment joining them. The C.G. of a uniform rod is at its middle point and that of a uniform circular disc is at its centre.

8.7 Analytical determination of the C.G.'s of some simple bodies

We shall now find out the positions of the C.G.'s of some simple bodies by using the formulas stated above. The integral formulas will be used wherever required and found more convenient.

8.8 C.G. of a thin rod

Let the length of the rod *OA* be 2a. We take *OA* as the x-axis. At a distance x from the end *O*, we take an elementary length $PQ = \delta x$ of the rod. We assume that the thickness of the rod is the same throughout its length and let *a* be the area of cross section of the rod. Hence, *a* is a constant.



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Case 1 : Let the rod be of uniform density.

Let ρ be its uniform density. Then the mass of the element *PQ* $= a\ \delta x.\ \rho$, since $a\ \delta x$ is the volume of *PQ*. Hence, $\delta m = a\ \delta x.\ \rho$.

Since this element of mass is at a distance $\bar{x}$ from *O*, the distance $\bar{x}$ of the C.G. of the rod is given by

$$\bar{x} = \frac{\int_0^{2a} a\ \rho\ x dx}{\int_0^{2a} a\ \rho\ dx} = \frac{\int_0^{2a} x dx}{\int_0^{2a} dx}.$$

$[\because\ a\ \rho$ is a constant$]$

The limits of integration are chosen so as to include the entire rod from the end $x=0$ to the end $x=2a$.

$$\text{Hence } \bar{x} = \frac{\left[\frac{x^2}{2}\right]_0^{2a}}{2a} = \frac{2a^2}{2a} = a$$

Hence, *the C.G. of a thin uniform rod of uniform density is at its middle point.*

Case 2 : Let the rod be of variable density.

Let the density of the rod be some function of x, say $\rho = f(x)$.

Then

$$\bar{x} = \frac{\int_0^{2a} a\ f(x)\ x dx}{\int_0^{2a} a\ f(x)\ dx} = \frac{\int_0^{2a} x\ f(x)\ dx}{\int_0^{2a} x\ f(x)\ dx},$$

In particular, if the density of the rod at any point is proportional to its distance from O, then

$f(x) = k\,x$, where k is a constant.

Hence,

$$\bar{x} = \frac{\int_0^{2a} kx^2\,dx}{\int_0^{2a} kx\,dx} = \frac{\left[\frac{x^3}{3}\right]_0^{2a}}{\left[\frac{x^2}{2}\right]_0^{2a}}$$

$$= \frac{\frac{8a^3}{3}}{\frac{4a^2}{2}} = \frac{4}{3}a.$$

Hence, *the centre of gravity of the rod lies at a distance* $\frac{4}{3}a$ *from the end O.*

Remark 1: In what follows, by 'a uniform rod', we shall mean a rod of uniform density and uniform cross-section unless otherwise stated so that the C.G, of such a rod should be taken always at its middle point.

Remark 2: A uniform wire can be treated as a very thin rod so that the C.G. of a uniform wire will also be at its middle point.

8.9 C.G.'s of several uniform rods forming closed regions

(*i*) *To find the C.G. of three uniform rods forming a triangle* (*or, the C.G. of a uniform wire bent into the form of a triangle*):

Let AB, BC and CA be three uniform rods jointed together to form a triangle ABC. Let the lengths of BC, CA and AB be a, b and c respectively. We assume that D, E and F are the middle points of the rods BC, CA and AB respectively and w is the weight per unit length of each of the rods. Since the rods are uniform, the weights of the three rods will be aw, bw and cw acting through their middle points D, E and F respectively. Wenow join DE, EF and FD. Now, the weight aw at D and cw at F can be combined into a weight $(a+c)$ w at P on FD such that $\frac{FP}{PD} = \frac{aw}{cw} = \frac{a}{c}$.

Now, $\frac{a}{c} = \frac{BC}{AB} = \frac{2EF}{2ED} = \frac{EF}{ED}$.

Hence, $\frac{FP}{PD} = \frac{EF}{ED}$,

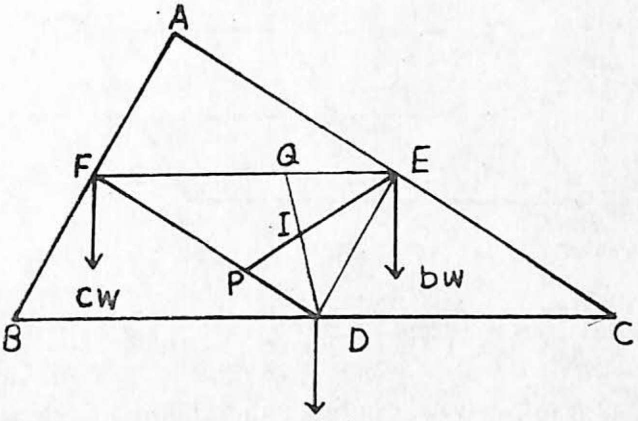

FIG. 123

which shows that EP is the bisector of $\angle FED$. If we now combine this weight $(b+c)$ w at P with the weight bw at E, then their resultant will pass through some point on EP. Hence, the C.G. of the three rods forming the triangle ABC lies at some point on the internal bisector of $\angle FED$.

Similarly, if we combine the weights at F and E first and then combine this resultant with the force at D, it will be found that the C.G. of the same three rods will lie at some point on DQ, the internal bisector of $\angle FDE$. Since the C.G. of a body is a unique point, the C.G. of the three rods must be at a common point situated on both EP and DQ i.e. the point, I, of intersection of these two bisectors. Clearly, the point, I, will be the in-centre of $\triangle DEF$. Hence, the required C.G. of the three rods will be at the in-centre of the triangle formed by joining the middle points of the rods.

Remark: The above example shows that the C.G. of a body or a system of bodies may not lie always in the body or bodies.

(ii) To find the C.G. of four uniform rods forming a parallelogram:

Let AB, BC, CD and DA be four uniform rods which form a parallelogram. Let P, Q, R and S be the middle points of AB, BC, CD and DA respectively. We join PR and SQ so as to intersec at G. From Geometry, it follows that G is the point of intersection of the two diagonals AC and BD of the parallelogram. Also PR and QS bisect each other at G.

Let $AB=DC=a$ and $AD=BC=b$ and let w be the weight per unit ength of each rod. Then the weights aw, aw of the rods AB and CD,

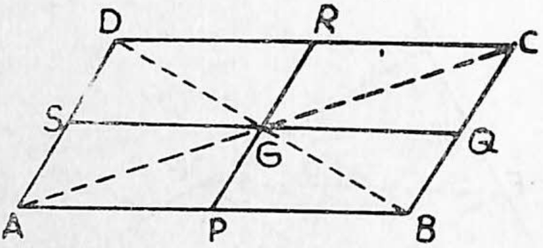

FIG. 124

acting at P and R respectively, can be combined into a single weight acting at G, the middle point of PR. Similarly, the weights bw, bw of the rods AD and BC, acting at S and Q respectively, can be conbined into a single force acting at G, the middle point of SQ. Hence, the total weight of the four rods acts at G. Hence, the required $C.G.$ of the four rods forming a parallelogram is at the point of intersection of its two diagonals.

8.10 C.G. of three equal weights placed at the vertices of a triangle

Let three equal weights each equal to w be placed at A, B, C of $\triangle ABC$. Let D, E and F be the middle points of BC, CA and AB respectively. The two equal weights w, w at B and C can be combined into a single force

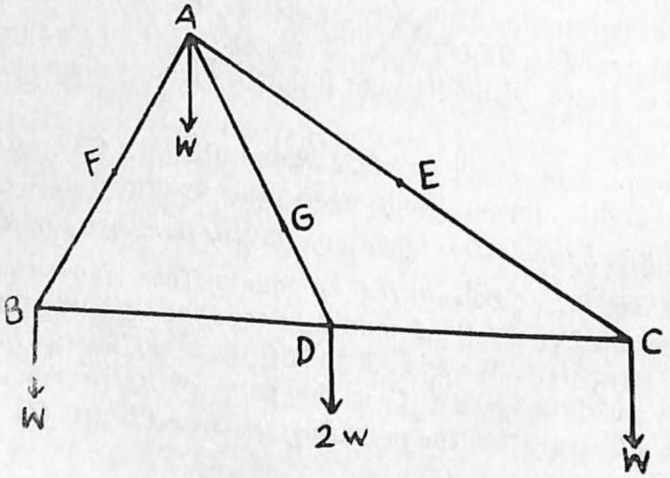

FIG 125

2w acting at D. If we now combine this weight with the weight at A, then we see that the resultant weight will act at G on AD such that $AG:GD = 2w : w = 2:1$. Hence, G must be the centroid of $\triangle ABC$. Hence, *the C.G. of three equal weights placed at the three vertices of a triangle lies at the centroid of the triangle.*

8.11 C.G. of a uniform triangular lamina

(A very thin tin plate or a card board may be considered as a lamina).

Let ABC be a uniform triangular lamina and D and E be the middle points of BC and CA respectively.

Let the medians AD and BE intersect at G.

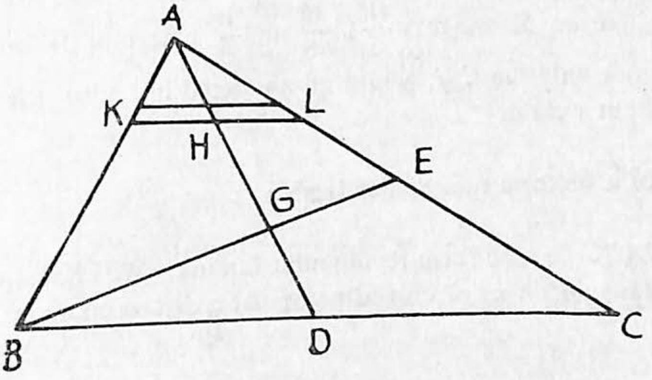

Fig 126

We now divide the triangle into an infinite number of thin strips by lines parallel to BC and let KL be any such strip. This strip can be treated as a thin uniform rod and so its C.G. will be at its middle point.

Let AD and KL intersect at H. Now, since $KHL \parallel BDC$, we have

$$\frac{KH}{BD} = \frac{AH}{AD} = \frac{HL}{DC}.$$

But since $BD = DC$, hence $KH = HL$ i.e. H is the middle point of KL.

Hence, the C.G. of the strip KL lies at H.

Similarly, the *C.G.* of each of other strips parallel to *BC* will lie on some point on *AD*. Hence, the *C.G.* of the entire lamina will be a point on the median *AD*. Similarly, dividing the triangle into an infinite number of thin strips each parallel to *AC*, we can show that the *C.G.* of the triangle *ABC* lies on some point on the median *BE*. Since the *C.G.* of a body is a unique point, hence the *C.G.* of the triangular lamina will be at the common point of intersection, *G*, of the medians *AD* and *BE* i.e. at the centroid of $\triangle ABC$. Thus *the C.G. of a uniform triangular lamina is at its centroid.*

Remark : We have seen that the *C.G.* of three equal weights placed at the three vertices of a triangle is at the centroid of the triangle. It follows that *the C.G. of a uniform triangular lamina is identical with that of any three particles of equal weights, placed at its vertices.*

We can, therefore, replace the weight *W* of a uniform triangular lamina by three equal weights $\frac{W}{3}$, $\frac{W}{3}$ and $\frac{W}{3}$ placed at its vertices, since in this case not only the *C.G.* would be unaltered but also the magnitude of the resultant weight.

8.12 C.G. of a uniform rectangular lamina

Let *OABC* be a uniform rectangular lamina. We chooss *OA* and *OC* as the rectangular axes of co-ordinates. At a distance x from *O*, we take

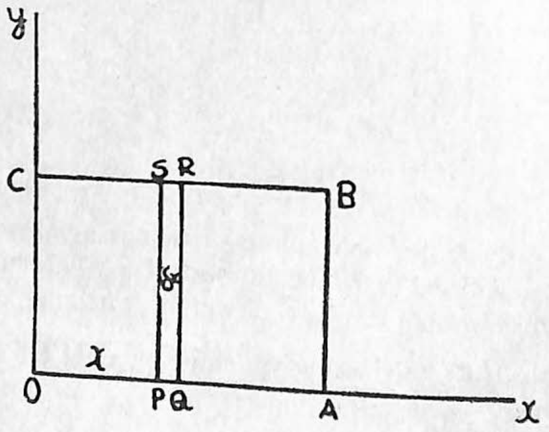

FIG. 127

a small elementary strip *PQRS* of breadth δx, drawn parallel to y-axis. Let ρ be the weight per unit area and the length and breadth of the rectangle be a and b respectively so that $OA=a$ and $OC=b$.

The area of $PQRS=b\delta x$.

Then the weight of the strip *PQRS* is $\delta w=b\delta x.\rho$.

The strip *PQRS* may be taken as a thin uniform rod so that the co-ordinates of its *C.G.* ard $\left(x,\ \frac{1}{2}b\right)$.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the *C.G.* of the rectangular lamina. Then

$$\bar{x}=\frac{\int x\,dw}{\int dw}=\frac{b\rho\int_0^a x\,dx}{b\rho\int_0^a dx}=\frac{\frac{1}{2}a^2}{a}=\frac{1}{2}a$$

and

$$\bar{y}=\frac{\int y\,dw}{\int dw}=\frac{b\rho\int_0^a \frac{1}{2}b\,dx}{b\rho\int_0^a dx}=\frac{\frac{1}{2}ba}{a}=\frac{1}{2}b\,.$$

Hence, *the C.G. of the rectangular lamina is at* $\left(\frac{1}{2}a,\ \frac{1}{2}b\right)$ *which is clearly the point of intersection of the two diagonals.*

8.1ε C.G. of a uniform parallelogram lamina

Let *ABCD* be uniform parallelogram lamina of weight *W*. Let the two diagonals *AC* and *BD* intersect at *O*. We know that $OA=OC$, $OB=OD$ and area of $\triangle ADC$=area of $\triangle ABC$.

$\therefore$ The weight of $\triangle ADC$ will be $\frac{W}{2}$ acting at the centroid G_1 of

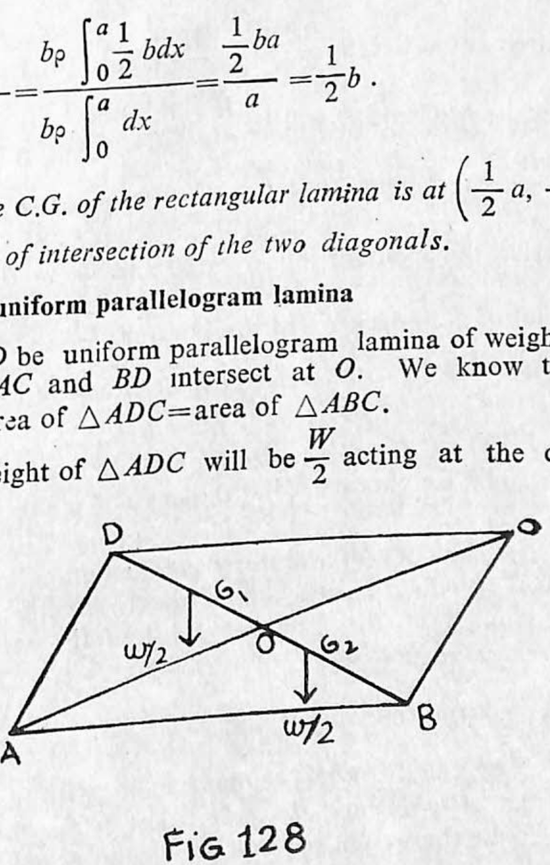

Fig 128

$\triangle ADC$ and that of $\triangle ABC$ is $\frac{W}{2}$ acting at the centroid G_2 of $\triangle ABC$. The resultant weight, W, will obviously pass through the middle point of $G_1 G_2$.

Now, $OG_1 = \frac{1}{3}\ OD$ and $OG_2 = \frac{1}{3}\ OB$ and since $OB = OD$, hence $OG_1 = OG_2$ so that the resultant weight W of the parallelogram will pass through O which is the point of intersection of the diagonals BD and AC. Hence, *the C.G. of a parallelogram lamina will be at the point of intersection of its two diagonals.*

An alternative method :

We replace the weight $\frac{W}{2}$ of $\triangle ADC$ by three equal weights $\frac{W}{6}$, $\frac{W}{6}$, $\frac{W}{6}$ placed at A, D and C. Similarly, the weight $\frac{W}{2}$ of $\triangle ABC$ can be replaced by three equal weights $\frac{W}{6}$, $\frac{W}{6}$, $\frac{W}{6}$ placed at A, B and C. The total weights at A, B, C and D will be $\frac{W}{3}$, $\frac{W}{6}$, $\frac{W}{3}$, $\frac{W}{6}$ respectively. The resultant of $\frac{W}{3}$ at A and $\frac{W}{3}$ at C is $\frac{2W}{3}$ acting at O, the middle point of AC. Similarly, the resultant of $\frac{W}{6}$ at D and $\frac{W}{6}$ at B will be $\frac{W}{3}$ at O, the middle point of BD. Hence, the total weight $\frac{2W}{3} + \frac{W}{3} = W$ of the lamina will pass through O, the point of intersection of the diagonals AC and BD.

Remark 1 : The C.G. of a uniform parallelogram lamina is the same as that of four equal weights placed at the angular points of the parallelogram.

Remark 2 : The C.G. of a uniform parallelogram lamina can also be found out directly by dividing the lamina into an infinite number of thin strips drawn parallel to the adjacent sides of the parallelogram as in section 8.11.

8.14 C.G. of a uniform trapezium lamina

Let $ABCD$ be a uniform trapezium lamina with parallel sides AB and DC and let $AB = 2a$ and $DC = 2b$. Let h be the height of the trapezium and w be the weight per unit area. Let F and E be the middle points of AB and CD respectively. We join AE, EB and EF. The trapezium

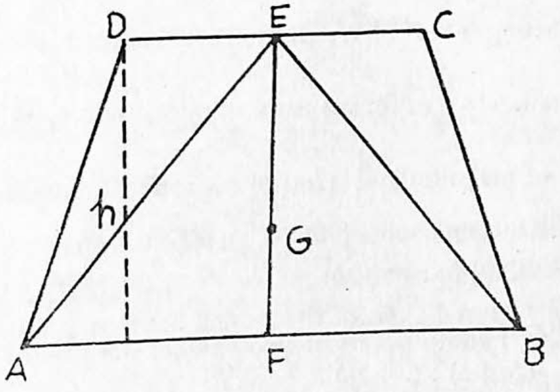

FIG. 129

consists of three triangles ADE, AEB and BEC of areas $\frac{1}{2}DE.\,h\mathrm{w}$, $\frac{1}{2}AB.\,h\mathrm{w}$ and $\frac{1}{2}EC.\,h\mathrm{w}$ i.e. $\frac{1}{2}\,bh\mathrm{w}$, $ah\mathrm{w}$ and $\frac{1}{2}\,bh\mathrm{w}$ respectively acting at their centroids. Now, the weight $\frac{1}{2}\,bh\mathrm{w}$ of $\triangle ADE$ can be replaced by three equal weights $\frac{1}{6}\,bh\mathrm{w}$, $\frac{1}{6}\,bh\mathrm{w}$ and $\frac{1}{6}\,bh\mathrm{w}$ acting at A, D and E. Similarly, the weight $\frac{1}{2}\,bh\mathrm{w}$ of $\triangle BEC$ can be replaced by three equal weights $\frac{1}{6}\,bh\mathrm{w}$, $\frac{1}{6}\,bh\mathrm{w}$ and $\frac{1}{6}\,bh\mathrm{w}$ at B, E and C, and the weight $ah\mathrm{w}$ of $\triangle AEB$ can be replaced by three equal weights $\frac{1}{3}\,ah\mathrm{w}$, $\frac{1}{3}\,ah\mathrm{w}$, $\frac{1}{3}\,ah\mathrm{w}$ at A, E and B. We thus get the total weight $\frac{1}{6}\,bh\mathrm{w}+\frac{1}{3}\,ah\mathrm{w}=\frac{(b+2a)\,h\mathrm{w}}{6}$ at each of A and B, $\frac{1}{6}bh\mathrm{w}$ at each of D and C and $\frac{1}{6}\,bh\mathrm{w}+\frac{1}{6}bh\mathrm{w}+\frac{1}{3}ah\mathrm{w}=\frac{1}{3}\,(a+b)\,h\mathrm{w}$ at E. The two equal weights at D and C can be combined together to give the resultant $\frac{1}{3}\,bh\mathrm{w}$ at E. Hence, the resultant of the weights at D, C

and E is $[\frac{1}{3}(a+b)+\frac{1}{3}b]\,h\mathrm{w}=\frac{1}{3}(a+2b)\,h\mathrm{w}$ acting at E. Also, the two equal weights at A and B can be combined together to give a resultant $\frac{(2a+b)\,h\mathrm{w}}{3}$ acting at F. We have thus reduced the weight of the trapezium lamina into twofor ces : one of magnitude $\frac{1}{3}(a+2b)\,h\mathrm{w}$ at E and the other of magnitude $\frac{1}{3}(2a+b)\,h\mathrm{w}$ at F. The resultant of these two forces will pass through some point G on EF such that

$$EG : GF=(2a+b) : (a+2b).$$

Thus the required C.G. of the lamina lies at a point G on the line EF joining the middle points of two parallel sides of the trapezium such that $EG : GF=(2a+b) : (a+2b)$.

8.15 C.G. of a uniform parabolic lamina bounded by a double ordinate

Let the equation of the parabolic lamina AOB be $y^2=4a\mathrm{x}$, where AB is the double ordinate whose equation is, say, $\mathrm{x}=b$. We take an elementary strip PQ of thickness $\delta\mathrm{x}$ parallel to y-axis at a distance x from O.

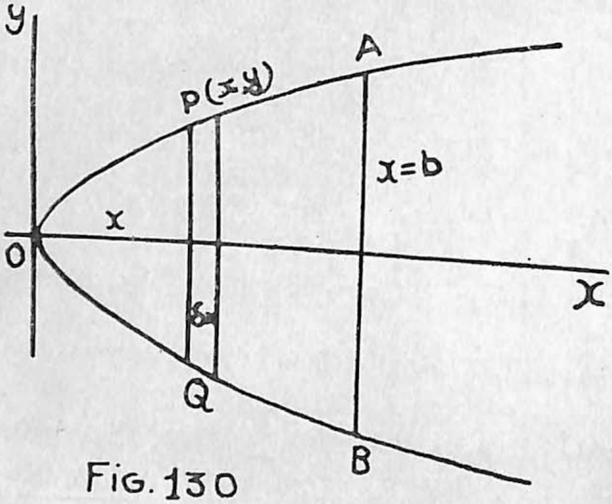

FIG. 130

Since all such strips are bisected by the x-axis, we see from symmetry that the C.G. of the lamina will lie on the x-axis. Let $\bar{\mathrm{x}}$ be the distance of the C.G. from O and let w be the weight per unit area of the lamina. Since the co-ordinates of P are (x, y), the area of the strip $PQ=2\mathrm{y}\,\delta\mathrm{x}$.

$$\therefore \text{ The weight of this strip}=2\mathrm{yw}\,\delta\mathrm{x}=2\sqrt{4\,a\mathrm{x}}\;\mathrm{w}\,\delta\mathrm{x}$$

$$=4\sqrt{a\mathrm{x}}\;\mathrm{w}\,\delta\,\mathrm{x}.$$

Hence,

$$\bar{x} = \frac{\int_0^b x.4\sqrt{ax}\,.\,wdx}{\int_0^b 4\sqrt{ax}\,.\,wdx}$$

$$= \frac{\int_0^b x^{3/2}\,dx}{\int_0^b x^{1/2}\,dx} = \frac{\frac{2}{5}\cdot\left[x^{5/2}\right]_0^b}{\frac{2}{3}\cdot\left[x^{3/2}\right]_0^b}$$

$$= \frac{3}{5}\,b.$$

Hence, *the required C.G. of the parabolic lamina is at a distance* $\frac{3}{5}$ *b from the vertex O.*

8.16 C.G. of a parabolic lamina bounded by the curve $y^2=4ax$, x-axis and the line $x=b$

We take an elementary strip *PQ*, parallel to y-axis, of thickness δx at a distance x from the origin *O*. *P* is the point (x, y), so that the area of *PQ* is $y\,\delta x$. If w be the weight per unit area, then the weight of this strip is $yw\,\delta x$.

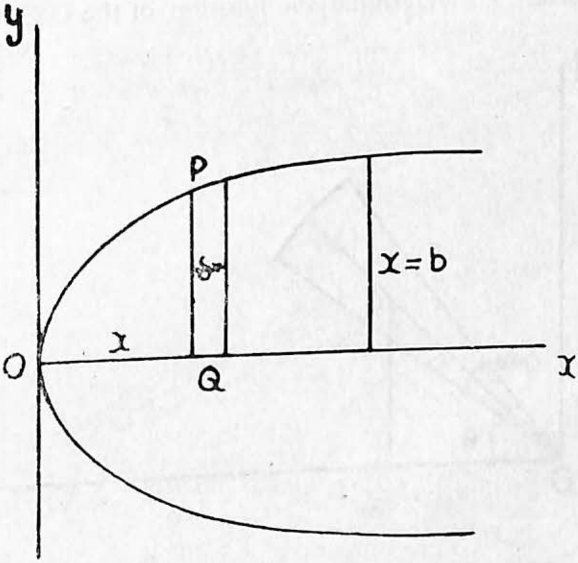

FIG. 131

The co-ordinates of the C.G. of this strip *PQ* which can be treated as a thin uniform rod are $\left(x, \frac{y}{2}\right)$. Let $(\bar{x}, \bar{y})$ be the required co-ordinates of the C.G. of the lamina,

$$\therefore \bar{x} = \frac{w\int_0^b xy\,dx}{w\int_0^b y dx} = \frac{\int_0^b x\sqrt{4ax}\,dx}{\int_0^b \sqrt{4ax}\,dx} = \frac{\int_0^b x^{3/2}\,dx}{\int_0^b x^{1/2}\,dx}$$

$$= \frac{3}{5}\,b \text{ [as in section 8.15]},$$

$$\bar{y} = \frac{w\int_0^b \frac{y}{2}\cdot y dx}{w\int_0^b y dx} = \frac{\frac{1}{2}\int_0^b 4ax\,dx}{\int_0^b \sqrt{4ax}\,dx}$$

$$= \frac{2a}{2\sqrt{a}}\cdot\frac{\left[\frac{x^2}{2}\right]_0^b}{\frac{2}{3}\cdot\left[x^{3/2}\right]_0^b} = \sqrt{a}.\ \frac{1}{2}\times\frac{3}{2}\cdot\sqrt{b}$$

$$= \frac{3}{4}\sqrt{ab}\ .$$

8.17 C.G. of a plane area bounded by a circular arc and two radii

We know, from symmetry, that the C.G. of a uniform circular plate is at its centre. To determine the position of the C.G. of a sector or a

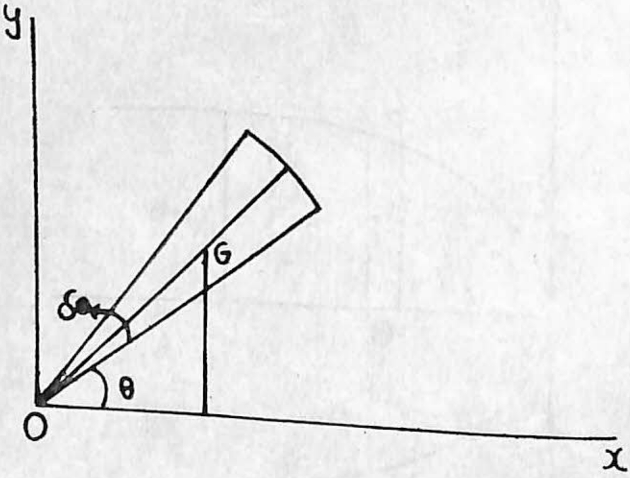

FIG.132

segment of a circle, we can conveniently introduce the polar co-ordinates (r,θ) whose relationship with the Cartesian co-ordinates (x,y) is given by the formulas

$$x=r\cos\theta,\ y=r\sin\theta.$$

We shall also use an important result for finding the area of the sector of a circle of radius r with the angle at the centre *O*, as $\delta\theta$. The area of this sector is given by $\frac{1}{2}r^2\,\delta\theta$. When $\delta\theta$ is very small, this sector can be treated approximately as a triangle so that the C. G. of this triangle will be at its centroid, *G*, whose Cartesian co-ordinates will be $\left(\frac{2}{3}r\cos\theta,\ \frac{2}{3}r\sin\theta\right)$ with sufficient degree of accuracy for our purpose [fig. 132.] We now determine the C.G.'s of such laminas.

(*i*) *C.G. of a uniform sector of a circle.*

Let *OACB* be a sector of a circle with centre *O* and let $\angle AOB=2\alpha$. Let r be the radius of the circle so that $OA=OC=OB=r$.

Let *OC* bisect $\angle AOB$. We take *OC* as the x-axis. From symmetry, we see that the *C.G.*, *G*, will lie on *O*x. Let the distance of C.G. of the sector from *O* be $\bar{x}$.

We consider a small triangular strip *OPQ* such that $\angle POQ=\theta$. If w be the weight per unit area, then the elementary weight of this strip is $\frac{1}{2}r^2\,\delta\theta.w$ and this weight acts at the point $\left(\frac{2}{3}r\cos\theta,\ \frac{2}{3}r\sin\theta\right)$ within the strip.

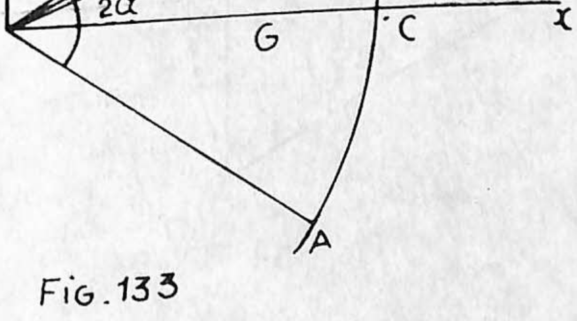

FiG. 133

$$\text{Hence, } \bar{x} = \frac{\int_{-\alpha}^{\alpha} \frac{2}{3} r \cos\theta . \ \frac{1}{2} r^2 \, d\theta}{\int_{-\alpha}^{\alpha} \frac{1}{2} r^2 \, d\theta}$$

$$= \frac{2}{3} r \frac{\int_{-\alpha}^{\alpha} \cos\theta \, d\theta}{\int_{-\alpha}^{\alpha} d\theta} \quad [\because r \text{ is a constant}]$$

$$= \frac{2}{3} r \frac{\Big[\sin\theta\Big]_{-\alpha}^{\alpha}}{\Big[\theta\Big]_{-\alpha}^{\alpha}} = \frac{2}{3} r . \ \frac{2\sin\alpha}{2\alpha}$$

$$= \frac{2}{3} r . \ \frac{\sin\alpha}{\alpha} .$$

Hence, *the C.G. lies on the bisector of* $\angle AOB$ *at a distance* $\frac{2}{3} r \frac{\sin\alpha}{\alpha}$ *from the centre O.*

Remark : For a semi-circle, $\alpha = \frac{\pi}{2}$ so that $OG = \frac{4a}{3\pi}$.

(ii) C.G. of a uniform segment of a circle.

Let *AOB* be a sector of a circle of radius r and centre *O*. Let *OC* be the bisector of $\angle AOB = 2\alpha$. We join *AB* to cut *OC* at *D*. We choose *OC* as the x-axis. We have to find the *C.G.* of the segment *ACBDA* of the

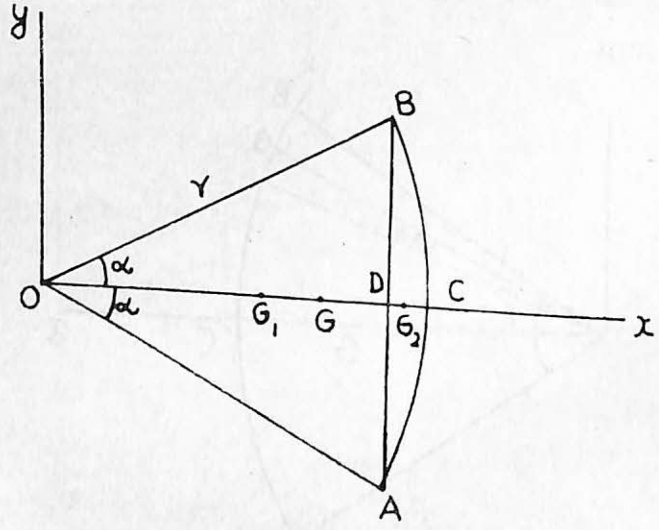

FIG 134

circle. From symmetry, the *C.G.* of the segment will lie on *OC*. Let $\bar{x}$ be the distance of its *C.G.*, G_2, from *O*. Let G_1 be the *C.G.* of $\triangle AOB$ and *G* be the *C.G.* of the sector *AOB*.

Since $OD = \text{r} \cos \alpha$,

$$\therefore OG_1 = \frac{2}{3} \text{r} \cos \alpha.$$

*A*s in (i),

$$OG = \frac{2}{3} \text{r}. \frac{\sin \alpha}{\alpha}.$$

Let w be the weight per unit area. Then the weight of $\triangle AOB = \frac{1}{2} AB.OD.\text{w} = \text{r}^2 \sin \alpha \cos \alpha.\text{w}$.

Area of the sector $AOB = \frac{1}{2} \text{r}^2.2\alpha = \text{r}^2\alpha$.

Hence, the weight of the sector $AOB = \text{r}^2\alpha\text{w}$.

Since the sector *AOB* consists of the triangle *AOB* and the segment *ACBDA*, the weight of the segment

$= \text{r}^2\alpha\text{w} - \text{r}^2 \sin \alpha \cos \alpha.\ \text{w}$.

Hence, the resultant of the weights $\text{r}^2 \sin \alpha \cos \alpha.\ \text{w}$ at G_1 and $(\text{r}^2\alpha - \text{r}^2 \sin \alpha \cos \alpha)$ w at G_2 is $\text{r}^2\alpha\text{w}$ passing through *G*.

Taking moments about *O*, we have

$$\text{r}^2 \sin \alpha \cos\alpha.\text{w}.OG_1 + (\text{r}^2\alpha - \text{r}^2 \sin \alpha \cos \alpha)\ \text{w}.OG_2 = \text{r}^2\alpha\ \text{w}.OG$$

$$\Rightarrow \text{r}^2 \sin \alpha \cos \alpha.\frac{2}{3}\text{r} \cos \alpha + (\text{r}^2\alpha - \text{r}^2 \sin \alpha \cos \alpha).\ OG_2 = \text{r}^2\alpha.\ \frac{2}{3}\text{r}.\ \frac{\sin \alpha}{\alpha}.$$

$$\text{Hence, } OG_2 = \frac{\frac{2}{3}\text{r}^3 \sin \alpha - \frac{2}{3}\text{r}^3 \sin \alpha \cos^2 \alpha}{\text{r}^2\alpha - \text{r}^2 \sin \alpha \cos \alpha}$$

$$= \frac{\frac{2}{3}\text{r}^3 \sin^2 \alpha}{\text{r}^2 (\alpha - \sin \alpha \cos \alpha)}$$

$$= \frac{2}{3}\text{r}. \frac{\sin^2 \alpha}{\alpha - \sin \alpha \cos \alpha}.$$

8.18 C.G. of a homogeneous solid right circular cone

We take the vertex *O* of the cone as the origin and the axis of the cone as the x-axis. At a distance x from *O*, we take an elementary circular strip *PQ* of thickness δx. The co-ordinates of *P* are (x, y). Let ρ be the density. Then the mass of his strip$= \pi \text{y}^2 \delta x \rho$.

From symmetry, the C.G. of the cone will be on its axis *OA*.

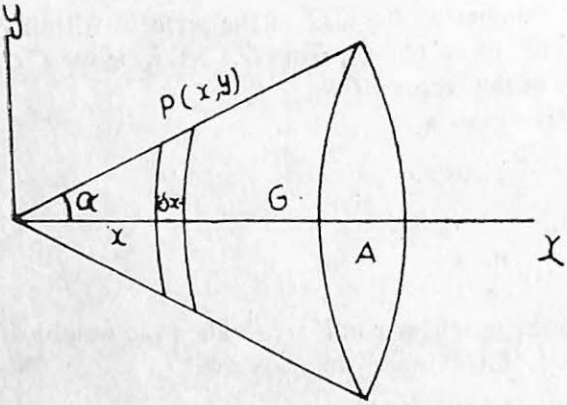

FiG. 135

Let the height *OA* of the cone be h and the semi-vertical angle be α. If the distance of C.G., *G*, of the cone from the vertex *O* be $\bar{x}$, then

$$\bar{x} = \frac{\int_0^h \pi y^2 \rho x dx}{\int_0^h \pi y^2 \rho dx} = \frac{\int_0^h x^3 \tan^2 \alpha \, dx}{\int_0^h x^2 \tan^2 \alpha \, dx}$$

$$\left[\because \frac{y}{x} = \tan \alpha\right]$$

$$= \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h} = \frac{3}{4} h.$$

Hence, *the C.G. of a cone lies on its axis at a distance* $\frac{3}{4}$ *h from its vertex, where h is its height.*

Remark : The C.G. of a solid right circular cylinder lies at the middle point of its axis and that of a sphere lies at its centre. These follow from the symmetry.

8.19 C.G. of a solid hemisphere

Let *O* be the centre of the sphere and *AB* be the base of the hemisphere. We take *OA* as the y-axis, and a line perpendicular to it through *O* as the x-axis. At a distance x from *O*, we take an elementary

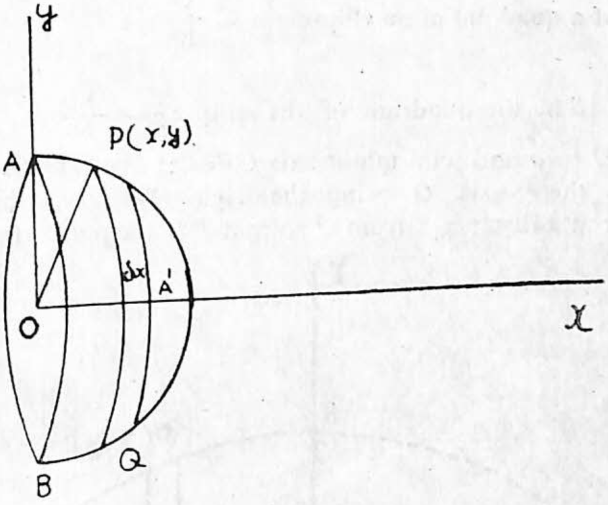

FIG. 136

circular strip *PQ* of thickness δx, where *P* is the point (x,y). If ρ be the density, then the weight of this strip is $\pi y^2 \, \delta x \, \rho$. Let *a* be the radius of the hemisphere. From symmetry, the C.G. of the hemisphere will lie on *O*x, a line through the centre *O* perpendicular to the base.

Now, let *PQ* intersect *O*x at *A'*

From $\triangle OPA'$ we have

$a^2 = x^2 + y^2 \Rightarrow y^2 = a^2 - x^2$.

If $\overline{x}$ be the distance of the C.G. of the hemisphere from *O*, then

$$\overline{x} = \frac{\int_0^a \pi y^2 \rho \, x \, dx}{\int_0^a \pi y^2 \rho \, dx} = \frac{\int_0^a x(a^2 - x^2)\, dx}{\int_0^a (a^2 - x^2)\, dx}$$

$$= \frac{\left[\frac{a^2 x^2}{2} - \frac{x^4}{4}\right]_0^a}{\left[a^2 x - \frac{x^3}{3}\right]_0^a} = \frac{\frac{a^4}{2} - \frac{a^4}{4}}{a^3 - \frac{a^3}{3}} = \frac{3}{8} a.$$

Hence, *the C.G. of a solid hemisphere lies on its radius perpendicular to its base at a distance of $\frac{3}{8}$ a from its base.*

8.20 C.G. of a quadrant of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let OAB be the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where semi-major axis $OA=a$ and semi-minor axix $OB=b$. We take OA as the x-axis and OB as the y-axis, O being the origin. We take a thin strip PQ of thickness δx at a distance x from O so that P is the point (x,y). If w be the weight per unit area, the elementary weight of PQ is $y.\,\delta x.w$ which acts at the middle point of PQ i.e. at the point $(x, \frac{y}{2})$. Let (x,y) be the co-ordinates of the C.G. of the quadrant of the ellipse. Then

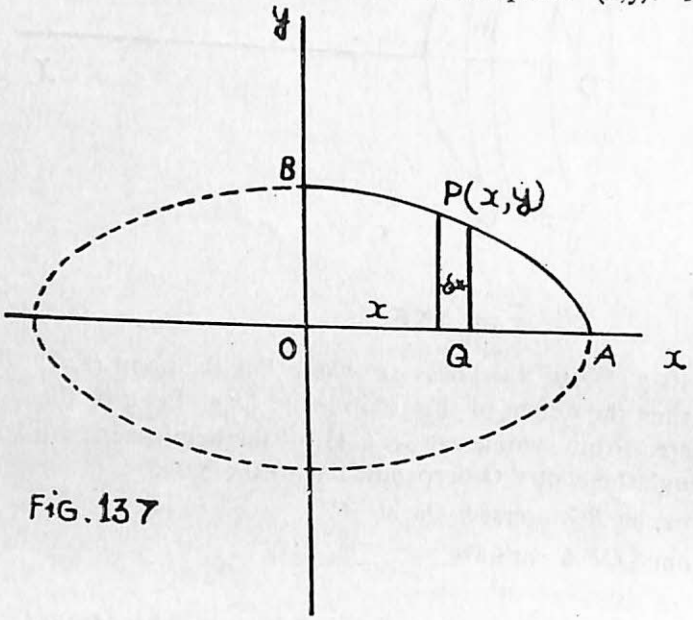

FIG. 137

$$\bar{x}=\frac{w\int_0^a xy\,dx}{w\int_0^a y\,dx}=\frac{\int_0^a x\,b\sqrt{1-\frac{x^2}{a^2}}\,dx}{\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx}$$

$$\left[\because\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right]$$

$$=\frac{\int_0^a x\sqrt{a^2-x^2}}{\int_0^a \sqrt{a^2-x^2}}\,.$$

Now, to evaluate $I_1 = \int_0^a x\sqrt{a^2 - x^2}\,dx$, we put

$$\sqrt{a^2 - x^2} = t$$

$$\Rightarrow a^2 - x^2 = t^2$$

$$\Rightarrow xdx = -\,t\,dt.$$

When $x=0$, $t=a$, and when $x=a$, $t=0$,

$$\therefore\ I_1 = \int_0^a t^2\,dt = \left[\frac{t^3}{3}\right]_0^a = \frac{a^3}{3}.$$

To evaluate $I_2 = \int_0^a \sqrt{a^2 - x^2}\,dx$,

we put $x = a\cos\theta$

$\therefore\ dx = -\,a\sin\theta\,d\theta.$

When $x = 0$, $\theta = \frac{\pi}{2}$ and when $x=a$, $\theta=0$.

$$\therefore\quad I_2 = a^2\int_0^{\pi/2} \sin^2\theta d\theta = \frac{a^2}{2}\int_a^{\pi/2} (1 - \cos 2\,\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\,\theta - \frac{\sin 2\,\theta}{2}\right]_0^{\pi/2} = \frac{\pi\,a^2}{4}.$$

$$\therefore\quad \bar{x} = \frac{\dfrac{a^3}{3}}{\dfrac{\pi\,a^2}{4}} = \frac{4\,a}{3\,\pi}.$$

$$\text{Similarly,}\ \bar{y} = \frac{w\displaystyle\int_0^a \frac{y}{2}\,.\,y\,dx}{w\displaystyle\int_0^a y\,dx}$$

$$= \frac{\dfrac{1}{2}\displaystyle\int_0^a\left(b^2 - \frac{b^2\,x}{a^2}\right)dx}{\dfrac{b}{a}\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx}$$

$$= \frac{\dfrac{1}{2}\left[b^2\,x - \dfrac{b^2\,x^3}{3a^2}\right]_0^a}{\dfrac{b}{a}\,I_2}$$

$$= \frac{\frac{1}{2}\left(b^2 a - \frac{b^2 a}{3}\right)}{\frac{\pi ab}{4}} = \frac{4b}{3\pi}.$$

Hence, the co-ordinates of the C.G. are

$$\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right).$$

8.21 Solved Examples

Ex. 1. D, E and F are the mid-points of the sides BC, CA and AB of the triangle ABC. Masses m_1, m_2, m_3 are placed at A, B, C respectively and masses K_1, K_2, K_3 are placed at D, E, F respectively. If the two systems have the same C.G., prove that

$$\frac{m_1}{K_2+K_3} = \frac{m_2}{K_3+K_1} = \frac{m_3}{K_1+K_2}.$$

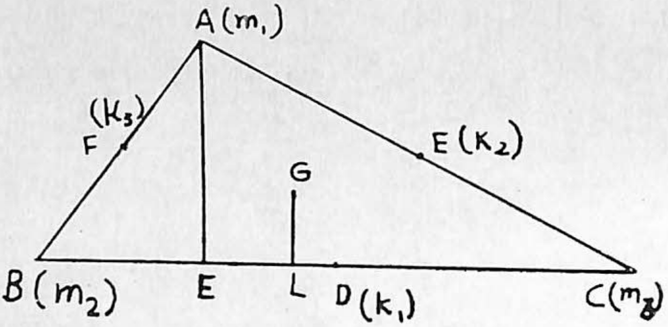

FiG. 138

We draw $AE \perp BC$. Let G be the common C.G. of the two systems and let GL be perpendicular to BC. Now, the mass K_1 at D is equivalent to $\frac{1}{2}K_1$ at B and $\frac{1}{2}K_1$ at a C, the mass K_2 at E is equivalent to $\frac{1}{2}K_2$ at A and $\frac{1}{2}K_2$ at C and the mass K_3 at F is equivalent to $\frac{1}{2}K_3$ at A and $\frac{1}{2}K_3$ at B. Hence, K_1, K_2 and K_3 at D, E and F are equivalent to $\frac{1}{2}(K_2+K_3)$ at A, $\frac{1}{4}(K_1+K_3)$ at B and $\frac{1}{2}(K_1+K_2)$ at

C. Since the resultant of weights proportional to m_1, m_2, m_3 passes through G, hence taking moments about BC, we have

$$(m_1+m_2+m_3)\ GL=m_1\ AE' . \qquad \ldots(1)$$

Again, since the resultant of the weights proportional to $\frac{1}{2}(K_2+K_3)$, $\frac{1}{2}(K_1+K_3)$ and $\frac{1}{2}(K_1+K_2)$ at A, B and C respectively passes through G, hence taking moments about BC, we get

$$(K_1+K_2+K_3)\ GL= \frac{1}{2} CK_2+K_3)\ AE' \qquad \ldots\ldots(2)$$

Dividing (1) by (2), we get

$$\frac{m_1+m_2+m_3}{K_1+K_2+K_3}=\frac{2m_1}{K_2+K_3}$$

$$\Rightarrow \frac{m_1}{K_2+K_3}=\frac{1}{2}\cdot\frac{m_1+m_2+m_3}{K_1+K_2+K_3}=\triangle, \text{ say.}$$

Similarly, taking moments about CA and AB, we get

$$\frac{m_2}{K_3+K_1}=\triangle$$

and $\dfrac{m_3}{K_1+K_2}=\triangle$.

Hence, $\dfrac{m_1}{K_2+K_3}=\dfrac{m_2}{K_3+K_1}=\dfrac{m_3}{K_1+K_2}$.

Ex. 2. A uniform wire is bent into the form of a triangle of sides of lengths a,b,c. Prove that the distances of the C.G. of the whole triangle from the sides are as

$$\frac{b+c}{a} : \frac{c+a}{b} : \frac{a+b}{c} .$$

[P. U. 1940]

Let ABC be the triangle with sides $BC=a$, $CA=b$ and $AB=c$. Let w be the weight per unit length of the wire. Then the weights aw, bw and cw of the rods BC, CA and AB act at their middle points D, E and F respectively. We can replace the weight aw at D by two equal weights $\frac{1}{2}a$w at B and $\frac{1}{2}a$w at C. Similarly, bw at E can be replaced by $\frac{1}{2}b$w at A and bw at C and cw at F can be replaced by $\frac{1}{2}c$w at A

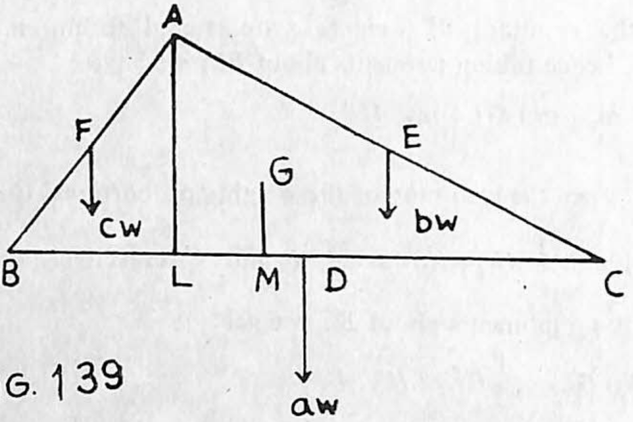

FIG. 139

and $\frac{1}{2}$ cw at B. Hence, the C.G. of the three wires is identical with the C.G. of three weights $\frac{1}{2}(b+c)\,w$ at A, $\frac{1}{2}(a+c)$ w at B and $\frac{1}{2}(a+b)$ w at C. Let $AL \perp BC$ and G be the C.G of the three wires. Let $GM \perp BC$ and $GM=x$. Taking moments about BC, we get

$$\frac{1}{2}(b+c)\,\text{w}.AL=(a+b+c)\ \text{w}.x$$

Now, $AL.\ BC=2\triangle$, where $\triangle$ is the area of $\triangle ABC$.

$$\therefore AL=\frac{2\triangle}{a}.$$

Hence, $\frac{1}{2}\ (b+c)\ \frac{2\triangle}{a}=(a+b+c)\ x$

$$\Rightarrow x=\frac{\triangle}{a+b+c}\cdot\frac{b+c}{a}=K.\ \frac{b+c}{a},\text{ where } K=\frac{\triangle}{a+b+c}.$$

If y and z be the distances of G from AB and AC respectively, then by taking moments about AB and AC, we get

$$y=K.\ \frac{c+a}{b}\ \text{and}\ z=K.\ \frac{a+b}{c}.$$

Hence, $x:y:z=\frac{b+c}{a}:\frac{c+a}{b}:\frac{a+b}{c}$.

Ex. 3. A triangle of uniform rods of different densities has its C.G at

(i) the circum-centre

(ii) the in-centre

of the triangle. Show that in the first case, the densities are proportional to sec A, sec B and sec C, and in the second case, they are proportional to

$\operatorname{cosec}^2 \dfrac{A}{2}$, $\operatorname{cosec}^2 \dfrac{B}{2}$ and $\operatorname{cosec}^2 \dfrac{C}{2}$.

(i) Let ρ_1, ρ_2 and ρ_3 be the densities of the rods BC, CA and AB respectively and let D, E and F be their middle points.

Let $BC=a$, $CA=b$ and $AB=c$. Then the masses $\rho_1 a$, $\rho_2 b$ and $\rho_3 c$ act at D, E and F respectively. Let O be the circum-centre of $\triangle ABC$. Then $OD \perp BC$. Let DO produced meet EF at L.

$\because EF=BC$, $\therefore OL \perp FE$. We join FD.

$\because FE \parallel DC$ and $FD \parallel EC$, the figure $FDCE$ is a parallelogram.

$\therefore \angle DFE=C$. $\therefore FL=FD \cos C=\dfrac{1}{2} b \cos C$.

Similarly, $LE=\dfrac{1}{2} c \cos B$.

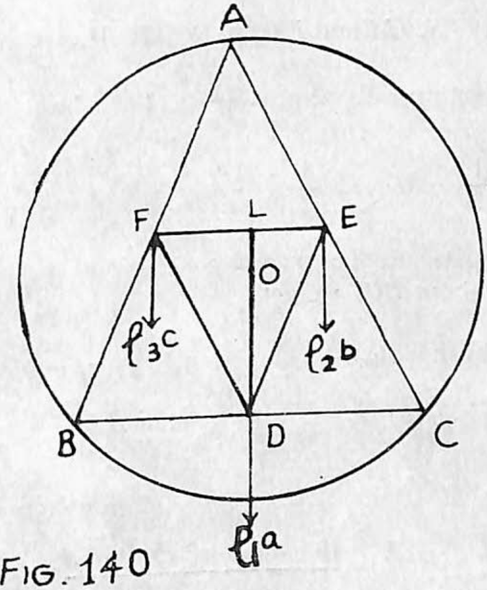

FIG. 140

Now, since the C.G. of the particles of masses $\rho_1 a$, $\rho_2 b$ and $\rho_3 c$ is at O, hence the C.G. of $\rho_2 b$ at E and $\rho_3 c$ at F must be at L.

$\because \rho_2 b : \rho_3 c = FL : LE = \dfrac{1}{2} b \cos C : \dfrac{1}{2} c \cos B$

$\Rightarrow \rho_2 : \rho_3 = \cos C : \cos B = \sec B : \sec C$.

Similarly, we get

$\rho_1 : \rho_2 = \sec A ; \sec B$.

Hence, $\rho_1 : \rho_2 : \rho_3 = \sec A : \sec B : \sec C$.

(ii) Let I be the in-centre of $\triangle ABC$; D, E and F are the middle points of BC, CA and AB respectively. From F, I and E, we draw

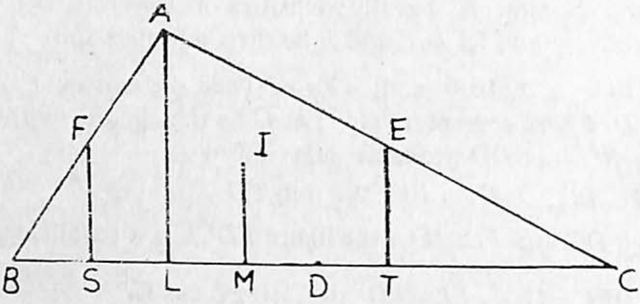

FIG 141

perpendiculars FS, IM and ET on BC. If $AL = p_1$ and $\triangle$ be the area of $\triangle ABC$, then $ap_1 = 2\triangle \Rightarrow p_1 = \dfrac{2\triangle}{a}$. Let r be the in-radius.

$$\text{Now, } FS = \frac{1}{2} \cdot AL = \frac{1}{2} \cdot \frac{2\triangle}{a} = \frac{\triangle}{a} = ET.$$

Taking moments of the masses $\rho_1 a$, $\rho_2 b$ and $\rho_3 c$ placed at D, E and F respectively about BC, we get

$$(\rho_1 a+\rho_2 b+\rho_3 c)\ r = (\rho_2 b+\rho_3 c)\ \frac{\triangle}{a} = \triangle \cdot \rho_2 \frac{b}{a} + \triangle \cdot \rho_3 \frac{c}{a}$$

$$= \triangle \cdot \rho_2 \frac{\sin B}{\sin A} + \triangle \cdot \rho_3 \frac{\sin C}{\sin A}$$

$$[\because a:b:c = \sin A : \sin B : \sin C]$$

$$\Rightarrow \rho_2 \sin B + \rho_3 \sin C = \frac{(\rho_1 a+\rho_2 b+\rho_3 c)\ r \sin A}{\triangle}$$

$$= k \sin A, \qquad \dots(1)$$

where $k = \dfrac{(\rho_1 a+\rho_2 b+\rho_3 c) r}{\triangle}$.

Similarly, taking moments about AB and AC, we get

$$\rho_3 \sin C + \rho_1 \sin A = K \sin B \qquad \dots(2)$$

$$\text{and } \rho_1 \sin A + \rho_2 \sin B = K \sin C. \qquad \dots(3)$$

Adding (2) and (3) and subtracting (1) from the sum, we get

$$\rho_1 \sin A=\frac{1}{2}k\,(\sin B+\sin C-\sin A)$$

$$=\frac{1}{2}k\left(2\sin\frac{B+C}{2}\cos\frac{B-C}{2}-\sin A\right)$$

$$=\frac{1}{2}k\left(2\cos\frac{A}{2}.\cos\frac{B-C}{2}-2\sin\frac{A}{2}.\cos\frac{A}{2}\right)\ [\because A+B+C=\pi]$$

$$=k\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)$$

$$=k\cos\frac{A}{2}.\,2\sin\frac{B}{2}.\sin\frac{C}{2}$$

$$=2k\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}.$$

$$\therefore \rho_1=\frac{k\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}=k\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ \operatorname{cosec}^2\frac{A}{2}.$$

Similarly, $\rho_2=k\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ \operatorname{cosec}^2\frac{B}{2}$

and $\rho_3=k\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ \operatorname{cosec}^2\frac{C}{2}$.

Hence, $\rho_1:\rho_2:\rho_3=\operatorname{cosec}^2\frac{A}{2}:\operatorname{cosec}^2\frac{B}{2}:\operatorname{cosec}^2\frac{C}{2}$.

Ex. 4. The longer side of a rectangle is double the shorter side and on the longer side an equilateral triangle is described. Find the C.G. of the lamina made up of the rectangle and the triangle.

Let $ABCD$ be a rectangle such that $AB=2AD$ and let DEC be an equilateral triangle. Let G_2 be the point of intersection of the two diagonals of the rectangle. Then G_2 will be the C.G. of the rectangular lamina. We join GE_2. Now, since $\triangle DEG_2\cong\triangle CEG_2$, hence $\triangle DEF\cong\triangle CEF$.

$\therefore \angle EFD=\angle EFC=90°$.

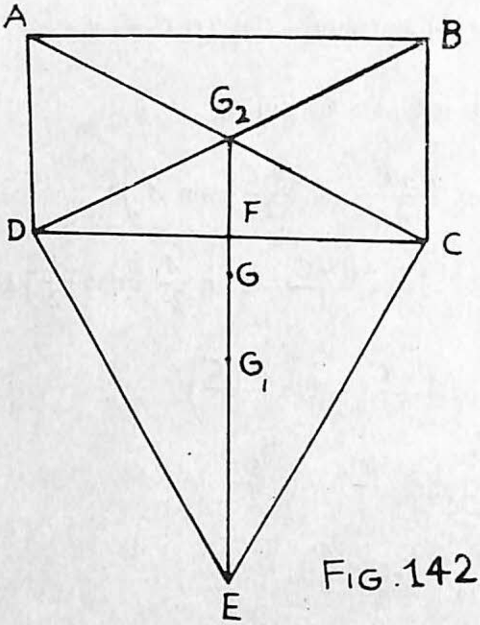

FIG. 142

Hence, EF must be a median of $\triangle DEC$. The C.G. of $\triangle DEC$ will, therefore, be at the centroid G_1 on EF.

Let $AD = a$. $\therefore DC = 2a = DE = EC$.

Then the area of $\triangle EDC = \frac{1}{2} \times 2a \times 2a \sin 60^0$

$$= 2a^2 . \frac{\sqrt{3}}{2} = \sqrt{3}\, a^2.$$

Area of the rectangle $ABCD = 2a^2$

Let w be the weight per unit area. Then the weight $2a^2$ w will act at G_2 and $\sqrt{3}\, a^2$ w will act at G_1. If G be the C.G. of the combined figure, then taking moment about BC, we get

$$2a^2 \text{ w}.\ G_2F - \sqrt{3}\, a^2 \text{w}.\ G_1F = (2a^2 + \sqrt{3}\, a^2) \text{ w}.\ GF$$

$$\Rightarrow 2.\frac{a}{2} - \sqrt{3}.\ \frac{1}{3}\ .\ 2a \sin 60^\circ = (2 + \sqrt{3}\,)\ GF$$

$$\Rightarrow a - a = (2 + \sqrt{3}\,)\ GF \Rightarrow GF = 0.$$

$\therefore$G will coincide with F.

Hence, the required C.G. lies at the middle point of the common side DC.

Ex. 5. A triangular lamina, obtuse-angled at C stands with the side AC in contact with a smooth table. Show that the least weight suspended from B, which will overturn the triangle, is

$$\frac{1}{3}\ W.\ \frac{a^2+3b^2-c^2}{c^2-a^2-b^2},$$

where W is the weight of the triangle. Interpret the above, if $c^2>a^2+3b^2$.

We replace the weight, W, of the lamina by three particles of weights $\frac{1}{3}$W, $\frac{1}{3}$W, $\frac{1}{3}$W placed at A, B, C.

Let w be the least weight suspended from B as a result of which the triangle ABC just turns about C. In this case, only the point C will be in contact with the table. Let R be the normal reaction of the table on the lamina.

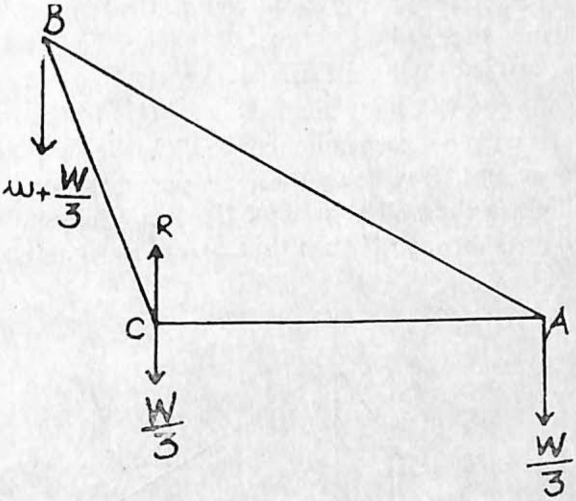

FIG. 143

Taking moments about C, we get

$$\frac{W}{3}.\ AC=\left(w+\frac{W}{3}\right)BC.\cos(\pi-C)$$

$$\Rightarrow\ -\cos C=-\frac{W.AC}{(3w+W)\ BC}$$

$$\Rightarrow\ \frac{c^2-a^2-b^2}{2ab}=\frac{Wb}{(3w+W)a}$$

$$\Rightarrow(3w+W)\ a\ (c^2-a^2-b^2)=2ab^2\ W$$

$$\Rightarrow 3wa\,(c^2-a^2-b^2)=W(2ab^2-ac^2+a^3+ab^2)$$
$$=W(3ab^2+a^3-ac^2)$$
$$\Rightarrow w=\frac{W}{3}\cdot\frac{a^2+3b^2-c^2}{c^2-a^2-b^2}\,.$$

If $c^2>a^2+3b^2$, w becomes negative and so the triangle will overturn about C due to its own weight only and there is no need of placing any additional weight at B. The significance of negative weight at B is that in order that the lamina may not overturn about C, the least force to be applied at B must be vertically upwards.

Ex. 6. Two uniform heavy rods, AB, BC rigidly united at B are hung up by the end A. Show that BC will be horizontal if $\sin C = \sqrt{2}\ \sin\frac{B}{2}$.

The weights of the two rods BC and AB will act at their middle points D and E respectively. For equilibrium, the tension T of the string at A must balance the rsultant vertical weight of the two rods. Let G be the C.G. of the two rods lying on ED. Hence, the line of action of T must pass through G and will be vertical. Since BC is horizontal, the vertical AM through A will be perpendicular to BC. We join AC. Let $BC=a$, $CA=b$ and $AB=c$ and let w be the weight per unit length of the rod. Then the resultant of the weight aw of BC acting at D and the weight cw at E of AB will pass through G such that $EG{:}GD=a{:}c$. Let $EN\perp BC$.

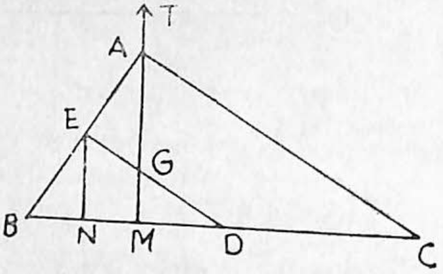

FIG 144

$$\text{Then } \frac{a}{c}=\frac{EG}{GD}=\frac{MN}{MD}=\frac{\frac{1}{2}BM}{BD-BM}=\frac{\frac{1}{2}c\cos B}{\frac{a}{2}-c\cos B}$$
$$=\frac{c\cos B}{a-2c\cos B}\,.$$

$$\Rightarrow a^2-2ac\cos B=c^2\cos B$$
$$\Rightarrow a^2-(c^2+a^2-b^2)=c^2\cos B$$
$$\Rightarrow b^2=c^2(1+\cos B)=c^2\,2\cos^2\frac{B}{2}$$
$$\Rightarrow \frac{b^2}{c^2}=2\cos^2\frac{B}{2}$$
$$\Rightarrow \frac{b}{c}=\sqrt{2}\cos\frac{B}{2}\left[\text{Taking the positive square root only.}\right]$$
$$\Rightarrow \frac{\sin B}{\sin C}=\sqrt{2}\cos\frac{B}{2}$$
$$\Rightarrow 2\sin\frac{B}{2}\cos\frac{B}{2}=\sqrt{2}\cos\frac{B}{2}\sin C$$
$$\Rightarrow \sin C=\sqrt{2}\sin\frac{B}{2}.$$

Hence the result.

Ex. 7. A triangular lamina is suspended successively from the angles A and B and the two positions of any side are found to be at right angles to each other. Prove that

$$a^2+b^2=5c^2.$$

Let G be the centroid of the triangle ABC and BE and AD be two medians. The weight of the lamina will pass through G vertically downwards and hence when hung from A, AG must be vertical. Similarly, when hung from B, BG will be vertical. Hence, the angle turned over by

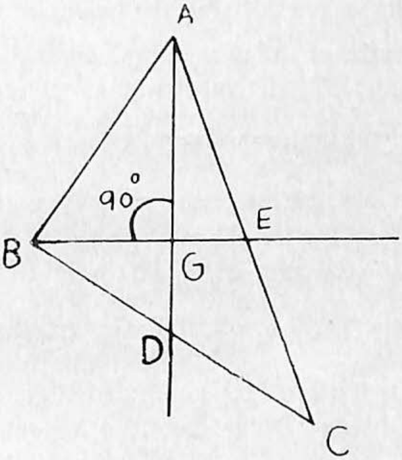

FIG 145

the lamina is $\angle AGB$. As the two positions of any side are at right angles, the angle turned over is 90°. Therefore, $\angle AGB=90°$.

Hence, $c^2=AB^2=AG^2+BG^2$

$$=\left(\frac{2}{3}AD\right)^2+\left(\frac{2}{3}BE\right)^2=\frac{4}{9}\left(AD^2+BE^2\right) \quad \ldots(1)$$

Now, $AB^2+AC^2=2(AD^2+BD^2)$ [$\because AD$ is a median.]

$$\Rightarrow c^2+b^2=2AD^2+2\left(\frac{1}{2}a\right)^2$$

$$\Rightarrow AD^2=\frac{1}{2}\left[c^2+b^2-\frac{a^2}{2}\right].$$

Similarly, since BE is a median of $\triangle ABC$,

$$BE^2=\frac{1}{2}\left(a^2+c^2-\frac{b^2}{2}\right).$$

Hence, from (1), $c^2=\frac{4}{9}\times\frac{1}{2}\left(c^2+b^2-\frac{a^2}{2}+a^2+c^2-\frac{b^2}{2}\right)$

$$\Rightarrow c^2=\frac{2}{9}\left(2c^2+\frac{b^2}{2}+\frac{a^2}{2}\right)$$

$$\Rightarrow 9c^2=4c^2+b^2+a^2 \Rightarrow a^2+b^2=5c^2.$$

Ex. 8. The centre of gravity of a hollow right circular cone, closed by a base, made of a thin uniform sheet, is the same as if the cone were solid. Prove that its vertical angle is $2\sin^{-1}\frac{1}{3}$. [Delhi 1964; Lucknow 1962]

[C.G. of a hollow right circular cone without base lies on its axis at *a* distance $\frac{2}{3}$ h from its vertex, h being the height of the cone.]

Let A be the vertex of the cone, h its height, l its slant height, r the radius of the base and 2θ the vertical angle of the cone. The area of the curved surface of the cone $=\pi\, rl=\pi r^2 \operatorname{cosec}\theta$ $\left[\because \frac{l}{r}=\operatorname{cosec}\theta\right]$

The area of the base of the cone $=\pi r^2$. If w be the weight per unit area of the surface, then the weight of the base of the cone is πr^2 w acting at O, the centre of the circular base. The weight of the curved surface of the hollow cone $=\pi r^2 \operatorname{cosec}\theta.$w acting at G_1 such that $AG_1=\frac{2}{3}\text{h}=r^2$.

If x be the distance of the combined C.G., G, of the hollow cone with the base, then taking moment about A, we get

$$(\pi r^2+\pi r^2\operatorname{cosec}\theta)\ \text{wx}=\pi r^2\ \text{hw}+\pi r^2\operatorname{cosec}\theta.\frac{2}{3}\text{hw}$$

$$\Rightarrow \text{x}=\frac{\pi r^2\text{h}\left(1+\frac{2}{3}\operatorname{cosec}\theta\right)}{\pi r^2(1+\operatorname{cosec}\theta)}$$

$$= \frac{h}{3} \cdot \frac{2+3\sin\theta}{1+\sin\theta}. \qquad \ldots(1)$$

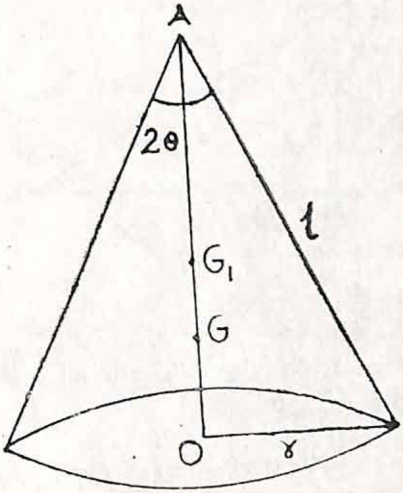

FIG. 146

For the solid cone, the distance of the C.G. from A is $\frac{3}{4}$ h. Since the two C.G.'s are given to be identical, hence $x = \frac{3}{4}h$.

Hence, from (1)

$$\frac{3}{4}h = \frac{h}{3} \cdot \frac{2+3\sin\theta}{1+\sin\theta}$$

$$\Rightarrow 9+9\sin\theta = 8+12\sin\theta$$

$$\Rightarrow 3\sin\theta = 1 \Rightarrow \theta = \sin^{-1}\frac{1}{3}.$$

Ex. 9. ABC is a thin uniform triangular lamina. Find the position of the point P within the triangle such that if the portion PBC be removed, the C.G. of the remainder is exactly at P. [C.U. 1963].

Let D be the middle point of BC so that AD is a median of $\triangle ABC$. Let P be any point within the triangle ABC. We join PB, PC and PD. The C.G. of $\triangle PBC$ must lie on its median PD. It is given that the C.G. of the remaining portion $ABPC$ is at P. Hence, the C.G. of $\triangle ABC$ must be on PD. But we know that the C.G. of $\triangle ABC$ lies on its median AD. Hence, PD must coincide with AD and the centroid G of $\triangle ABC$ must

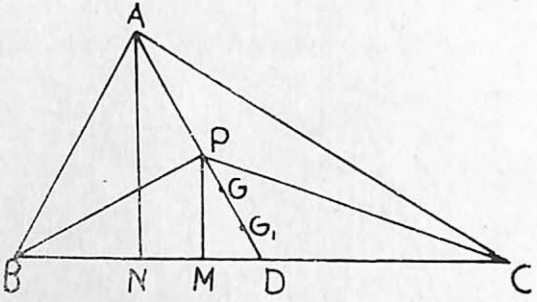

FIG. 147

lie on PD. Let G_1 be the centroid of $\triangle PBC$ and let w be the weight per unit area. Let $PM \perp BC$ and $AN \perp BC$.

$$\text{Now, } \frac{\text{area of } \triangle PBC}{\text{area of } \triangle ABC}=\frac{\frac{1}{2}BC.PM}{\frac{1}{2}BC.AN}=\frac{PM}{AN}=\frac{PD}{AD}.$$

Hence, area of $\triangle PBC=\dfrac{PD}{AD}\,\Delta$, where Δ is the area of $\triangle ABC$.

Hence, the area of the remaining portion $ABPC$

$$=\Delta-\frac{PD}{AD}\Delta=\frac{(AD-PD)\Delta}{AD}=\frac{AP.\Delta}{AD}.$$

Hence, taking moments about A, we get

weight of $\triangle PBC \times AG_1+$weight of $ABDC \times AP=$weight of $\triangle ABC \times AG$

$$\Rightarrow \frac{PD}{AD}\cdot \Delta \text{w} \times \left(AP+\frac{2}{3}PD\right)+\frac{AP}{\text{AD}}\cdot \Delta \text{w}.\, AP$$

$$=\Delta \text{w}.\,\frac{2}{3}\left(AP+PD\right)$$

$$\Rightarrow PD.AP+\frac{2}{3}PD^2+AP^2=\frac{2}{3}AP.AD+\frac{2}{3}AD.PD$$

$$=\frac{2}{3}AP\left(AP+PD\right)+\frac{2}{3}PD\left(AP+PD\right)$$

$$=\frac{2}{3}AP^2+\frac{2}{3}AP.PD+\frac{2}{3}PD.AP+\frac{2}{3}PD^2.$$

$$\Rightarrow -\frac{1}{3}PD.AP+\frac{1}{3}AP^2=0$$

$$\Rightarrow PD=AP.$$

Hence, P will be the middle point of the median AD.

Ex. 10. A pack of cards is laid on a table, and each card projects in the direction of the length of the pack beyond the one below it. If each projects as far as possible, show that the distance between the extremities of successive cards from the top will form a harmonic progression.

Let A_1, A_2, A_3, A_4......be the extreme ends of the successive cards from the top and let G_1, G_2, G_3, G_4 ..be their respective C.G.'s. Let $2l$ be the lengths and w be the weight of each card. The C.G. of the topmost card i.e. G_1 must, in the extreme case, be just supported at A_2, otherwise it would topple over. Hence, it can at most project a length l i.e. the horizontal distance $A_1 A_2 = l$. Similarly, the combined C.G. of the two topmost cards must, in the extreme case, be vertically above A_3. In other words, the resultant of w at G_1 and w at G_2 must pass through A_3. Hence, w × $A_2 A_3$ = w × horizontal distance between G_2 and A_3

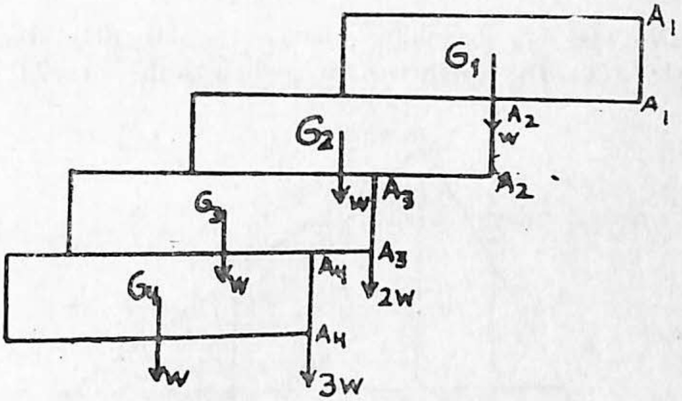

FIG. 148

$$= w(l - A_2A_3)$$

$$\Rightarrow 2A_2A_3 = l \Rightarrow A_2A_3 = \frac{l}{2}.$$

Again, the combined C.G. of the three topmost cards must in the extreme case, be vertically over A_4 i.e. the resultant of 2w acting at A_3 and w at G_3 must pass through A_4.

Hence, 2w × $A_3 A_4$ = w × horizontal distance between G_3 and A_4

$$\Rightarrow 2w \times A_3 A_4 = w(l - A_3A_4)$$

$$\Rightarrow 3A_3 A_4 = l \Rightarrow A_3A_4 = \frac{l}{3},$$

and so on.

Hence, the successive distances between the extremities of the cards are $l, \frac{l}{2}, \frac{l}{3}, \frac{l}{4}$...... which form a H.P.

Ex. 11. A piece of wire is formed into a triangle *ABC*. Show that if x, y, z be the distances of the C.G. of the wire from the three sides of the triangle, then

$$4xyz-r^2(x+y+z)=r^3,$$

where r is the radius of the in-circle of the triangle.

Let *D*, *E* and *F* be the middle points of the sides *BC*, *CA* and *AB* of the triangle *ABC*. The weights *a*w *b*w, and *c*w of the wires *BC*, *CA* and *AB* act through *D*, *E* and *F* respectively, where w is the weight of the wire per unit length and *a*, *b*, *c* are the lengths of the sides of the triangle.

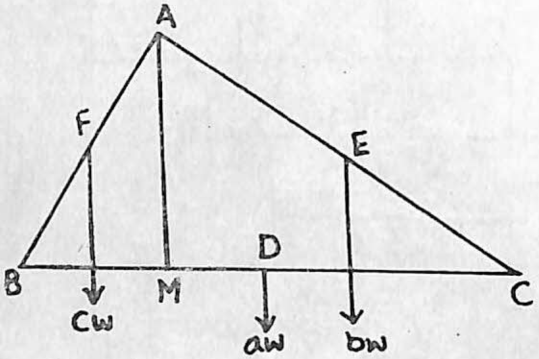

FIG. 149

Let $AM \perp BC$. $\therefore AM = c \sin B = b \sin C$.

Hence, the distances of *F* and *E* from *BC* are respectively

$\frac{1}{2} c \sin B$ and $\frac{1}{2} b \sin C$.

Taking moments about *BC*, we have

$$x(a+b+c)\,w = bw.\ \frac{1}{2} b \sin C + cw.\ \frac{1}{2} c \sin B$$

$$\Rightarrow x = \frac{\frac{1}{2} b^2 \sin C + \frac{1}{2} c^2 \sin B}{a+b+c}$$

$$=\frac{\frac{b}{a}\cdot\frac{1}{2}ab\sin C+\frac{c}{a}\cdot\frac{1}{2}ac\sin B}{a+b+c}$$

$$=\frac{\frac{b}{a}\triangle+\frac{c}{a}\triangle}{a+b+c}\text{, where }\triangle=\text{area of }\triangle ABC$$

$$=\frac{1}{a}\cdot\frac{(b+c)\triangle}{2\mathrm{s}}\text{ , where }2\mathrm{s}=a+b+c.$$

Now, $\mathrm{r}=\frac{\triangle}{\mathrm{s}}\Rightarrow\triangle=\mathrm{rs}.$

Hence, $\mathrm{x}=\frac{(b+c)\mathrm{rs}}{2a\mathrm{s}}=\frac{\mathrm{r}}{2a}(b+c).$

Similarly, $\mathrm{y}=\frac{r}{2b}(c+a)$ and $z=\frac{r}{2c}(a+b).$

Now, $4\mathrm{xy}z-\mathrm{r}^2(\mathrm{x}+\mathrm{y}+z)$

$$=\frac{4\mathrm{r}^3}{8abc}(b+c)(c+a)(a+b)-\mathrm{r}^2.\ \frac{\mathrm{r}}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

$$=\frac{\mathrm{r}^3}{2abc}[(bc+ab+c^2+ca)(a+b)-bc\,(b+c)-ca(c+a)-ab(a+b)]$$

$$=\frac{\mathrm{r}^3}{2abc}(abc+a^2b+ac^2+a^2c+b^2c+b^2a+bc^2+abc-b^2c-bc^2-c^2a$$
$$-ca^2-a^2b-ab^2)$$

$$=\frac{\mathrm{r}^3}{2abc}\times 2abc=r^3.$$

Hence the result.

Ex. 12. A circular disc of radius 1m has two equal circular holes punched in it, the centres of these holes being at the middle points of two radii of the disc at right angles to one another. If the radius of each hole is $\frac{1}{6}$m, find the distance of the C. G. of the remainder from the centre of the disc.

Let the centre of the circular disc be at O and let OA and OB be its two perpendicular radii. Let G_1 and G_2 be the centres of the two holes on

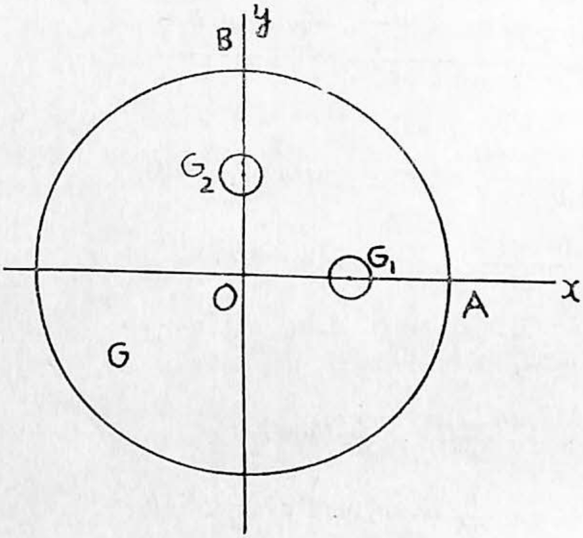

FiG. 150

OA and *OB* respectively such that $OG_1=G_1A=\frac{1}{2}$ and $OG_2=G_2B=\frac{1}{2}$. We take *OA* and *OB* as the x and y-axes respectively.

The area of each of the holes$=\pi\left(\frac{1}{6}\right)^2=\frac{\pi}{36}$ and the total area of the circular disc$=\pi 1^2=\pi$. Hence, the area of the remaining part of the disc when the holes are made

$$=\pi-\frac{\pi}{18}=\frac{17\pi}{18}.$$

If w be the weight per unit area, then the weights of the circular discs removed are $\frac{\pi w}{36}$ at G_1, $\frac{\pi w}{36}$ at G_2 and the weight of the remaining part of the disc is $\frac{17\pi w}{18}$. The total weight of the disc (without the holes)$=\pi w$ which acts at *O*.

Let the weight of the remaining part of the disc act at *G* (x_1, y_1). since the resultant of three forces acting at G_1, *G* and G_2 passes through *O*, taking moments about *O*x, we get

$$\frac{\pi w}{36}\times\frac{1}{2}+\frac{17\pi w}{18}x_1=0$$

$$\Rightarrow x_1 = -\frac{\pi w}{72} \times \frac{18}{17\pi w} = -\frac{1}{68}.$$

Taking moments about y-axis, we get

$$\frac{\pi w}{36} \times \frac{1}{2} + \frac{17\pi w}{18} y_1 = 0 \Rightarrow y_1 = -\frac{1}{68}.$$

Hence, the co-ordinates of the required C.G. are $\left(-\frac{1}{68}, -\frac{1}{68}\right)$ at a distance $\sqrt{\frac{1}{68^2} + \frac{1}{68^2}} = \frac{1}{68}\sqrt{2}$ from the centre of the circular disc.

Ex. 13. Prove that if the radius of one plane end-face of a frustum of a rigid circular cone is n times the radius of the other end-face, the C.G. divides the axis of the frustum in the ratio

$$(3n^2+2n+1) : (n^2+2n+3).$$

Let *ABCD* be the frustum of the cone *OAB* with axis *OE*, *O* being the vertex. Let *AB* be a diameter of the base of the cone. Let G_1, G_2 and G be the C.G.'s of the smaller cone *ODC*, the frustum and the whole cone *OA*B respectively all lying on the axis *OE*. Let the radius *CF* of the base of the smaller cone be r so that the radius *BE* of the bigger cone is nr. If the height *OF* of the smaller cone be h, then

$$\frac{h}{OE} = \frac{FC}{EB} = \frac{1}{n} \Rightarrow OE = nh.$$

$\therefore$ Volumes of the smaller and bigger cones are respecrively $\frac{1}{3}\pi r^2 h$ and $\frac{1}{3}\pi n^2 r^2.\ nh = \frac{1}{3}\pi n^3 r^2 h$.

Hence, the volume of the frustum

$$= -\frac{1}{3}\pi r^2 h + \frac{1}{3}\pi r^2 n^3 = \frac{1}{3}\pi r^2 h\left(-1+n^3\right).$$

If w be the weight per unit volume, then the weight of the cone *ODC* is $\frac{1}{3}\pi r^2 hw$ acting at G_1, that of the frustum is $\frac{1}{3}\pi r^2 h\left(n^3-1\right)w$ acting at G_2, while the weight of the whole cone is $\frac{1}{3}\pi r^2 hn^3$ w which acts at *G*. Hence, taking moments about *O*, we get

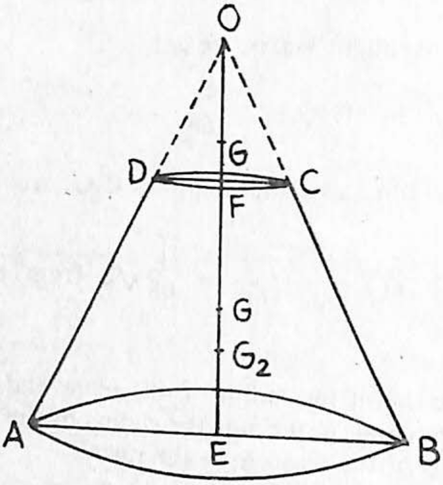

FiG. 151

$$\frac{1}{3}\pi r^2hw.OG_1+\frac{1}{3}\pi r^2h(n^3-1)w.OG_2=\frac{1}{3}\pi r^2hn^3.OG$$

$$\Rightarrow OG_1+(n^3-1)OG_2=n^3.OG$$

$$\Rightarrow \frac{3}{4}h+(n^3-1)\,OG_2=n^3.\frac{3}{4}nh$$

$$\Rightarrow OG_2=\frac{\frac{3}{4}n^4h-\frac{3}{4}h}{n^3-1}=\frac{3}{4}h.\,\frac{n^4-1}{n^3-1}$$

$$\therefore FG_2=OG_2-OF=\frac{3}{4}h.\,\frac{n^4-1}{n^3-1}-h$$

$$=\frac{3n^4-3-4n^3+4}{4(n^3-1)}\,.\,h$$

$$=\frac{3n^4-4n^3+1}{4(n^3-1)}.\,h$$

Also, $EG_2=OE-OG_2=nh-\frac{3h(n^4-1)}{4(n^3-1)}$

$$= \frac{4n^4-4n-3n^4+3}{4(n^3-1)}h$$

$$= \frac{n^4-4n+3}{4(n^3-1)}\,h\,.$$

Hence, $$\frac{FG_2}{EG_2} = \frac{3n^4-4n^3+1}{n^4-4n+3} = \frac{(n-1)^2\,(3n^2+2n+1)}{(n-1)^2\,(n^2+2n+3)}$$

$$= \frac{3n^2+2n+1}{n^2+2n+3}\,.$$

Ex. 14. Find the centroid of the area between $y^2=x$ and $y=x$.

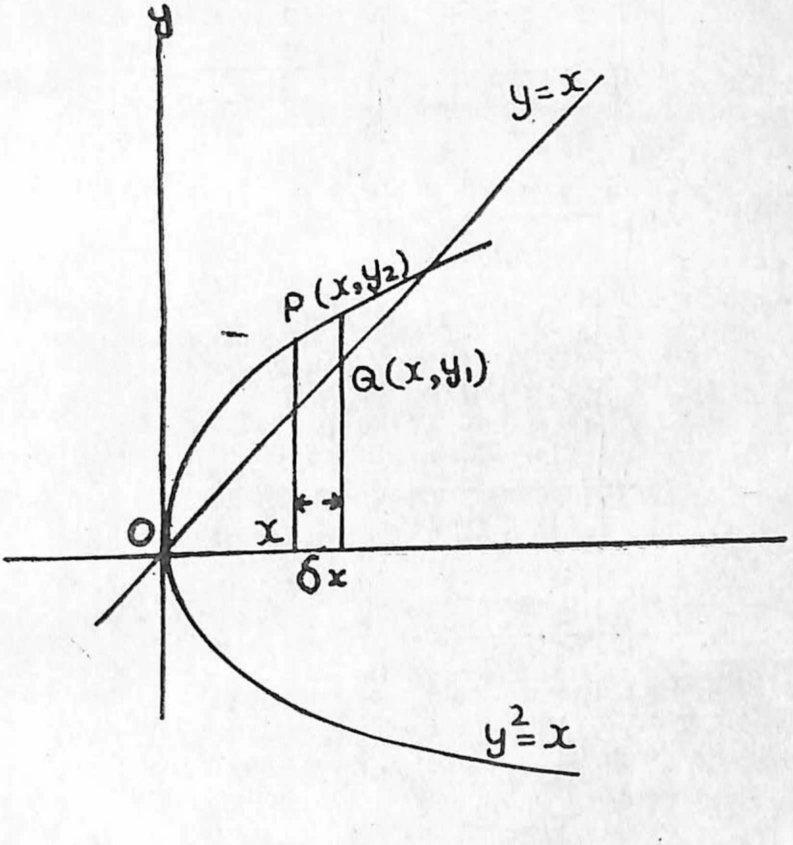

Fig. 15

Let the two curves intersect at $O(0, 0)$ and A whose co-ordinates are (1,1). We take an elementary strip PQ of thickness δx at a distance x from O, where *PQ* is parallel to y-axis. Let the co-ordinates of *P* and *Q* be (x,y_2) and (x,y_1) respectively and let $(\bar{x}, \bar{y})$ be the co-ordinates of the required centroid. The co-ordinates of the centroid of *PQ* will be $\left(x, \frac{y_1+y_2}{2}\right)$.

$$\text{Hence, } \bar{x}=\frac{\int_0^1 x(y_2-y_1)\,dx}{\int_0^1 (y_2-y_1)\,dx}$$

$$=\frac{\int_0^1 x(\sqrt{x}-x)\,dx}{\int_0^1 (\sqrt{x}-x)\,dx}, \quad [\text{since } y_1=x \text{ and } y_2=\sqrt{x}]$$

$$=\frac{\left[\frac{2}{5}x^{\frac{5}{2}}-\frac{1}{3}x^3\right]_0^1}{\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}x^2\right]_0^1}=\frac{\frac{2}{5}-\frac{1}{3}}{\frac{2}{3}-\frac{1}{2}}=\frac{2}{5}.$$

$$\text{Also, } \bar{y}=\frac{\int_0^1 \frac{1}{2}(y_1+y_2)(y_2-y_1)dx}{\int_0^1 (y_2-y_1)\,dx}$$

$$=\frac{\frac{1}{2}\int_0^1 (y_2^2-y_1^2)\,dx}{\int_0^1 (y_2-y_1)\,dx}$$

$$=\frac{1}{2}\frac{\int_0^1 (x-x^2)\,dx}{\int_0^1 (\sqrt{x}-x)\,dx}$$

$$=\frac{1}{2}\frac{\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1}{\left[\frac{2}{3}x^3-\frac{1}{2}x^2\right]_0^1}$$

$$=\frac{1}{2}\frac{\frac{1}{2}-\frac{1}{3}}{\frac{2}{3}-\frac{1}{2}}$$

$$=\frac{1}{2}.$$

Hence, the required co-ordinates of the centroid are $\left(\frac{2}{5}, \frac{1}{2}\right)$.

EXERCISE 8.1

1. *A* uniform rod *AB*, of length 4m and weight $6N$ has weights attached to it as follows : $6N$ at A, $2N$ at 1m from A, $3N$ at 2m from A, $4N$ at 3m from A and $5N$ at B. Find the distance of their *C.G.* from *A*.

[Ans. 2m]

2. *A* unifrom wire *AOB* is bent into two straight portions, *OA* and *OB*, inclined at an angle θ, *OA* and *OB* being of lengths a and b respectively. If the portion *OB* is horizontal when the whole is supported at *A*, prove that

$$\cos \theta = \frac{b^2}{a(a+2b)}.$$

3. A uniform rod is bent at right angles into two parts of lengths a and b. If the rod is suspended in equilibrium from the end of the arm of length a, prove that the inclination θ of the arm to the horizontal is given by

$$\tan \theta = \frac{a^2}{b^2} + \frac{2a}{b}.$$

4. *AB* and *AC* are two uniform rods of lengths $2a$ and $2b$ respectively. If $\angle BAC = \theta$, prove that the distance from *A* of the *C.G.* of the two rods is

$$\frac{(a^4 + 2a^2b^2 \cos \theta + b^4)^{1/2}}{a+b}.$$ [C. U. 1939]

5. Find the locus of the *C. G.* of triangle whose base is fixed and (i) whose vertical angle is given, (ii) whose vertex moves on a given straight line.

[Ans. (i) a circle, (ii) a straight line parallel to the given straight line.]

6. Three rods of unequal lengths are joined together to form *a* triangle *ABC*. If the masses are equal, prove that the *C. G.* coincides with that of the area. If the masses of the sides a, b, c are proportional to $b+c-a$, $c+a-b$, $a+b-c$, prove that tne *C. G.* is the in-centre.

7. A uniform wire 24m long is bent into the shape of a triangle *ABC*, the sides *BC*, *CA* and *AB* being as 3: 4: 5. Particles of weights p, q and r are placed at *A*, *B* and *C* respectively and it is found that the *C. G.* is unchanged. Prove that

$$p : q : r = 9 : 8 : 7.$$

8. A thin uniform wire is bent into the form of a triangle ABC and heavy particles of weights P, Q, R are placed at the angular points. Prove that if the $C.G.$ of the particles coincides with that of the wire, then

$$\frac{P}{b+c}=\frac{Q}{c+a}=\frac{R}{a+b},$$

where v, b, c are the lengths of the sides BC, CA and AB respectively.

9. Three like parallel forces P, Q and R act at the corners of a triangle. If their centre is the $C. G.$ of the triangle, prove that $P=Q=R$.

10. A thin uniform wire is bent into a triangle ABC; prove that its $C. G.$ is the same as that of three weights $\frac{b+c}{2}$, $\frac{c+a}{2}$ and $\frac{a+b}{2}$ placed at A, B, C respectively, where a, b and c are the lengths of the sides BC, CA and AB respectively. [C. U. 1946]

11. The in-circle of a triangle ABC touches the sides BC, CA and AB in D, E and F respectively. Prove that the $C. G.$ of the weights proportional to BC, CA and AB and placed at A, B and C respectively coincides with the $C. G.$ of the same weights placed at D, E and F respectively.

12. What is the form of a triangle if its $C. G.$ coincides with the centre of the circle circumscribing it ? [Ans. Equilateral]

13. Find the $C. G.$ of three particles of equal weights placed at the three angular points of a right-angled isosceles triangle whose hypotenuse is equal to 8 units. $\left[\text{Ans. } 2\frac{2}{3} \text{ units from the right angle.}\right]$

14. An isosceles right-angled triangle has squares described externally on all its sides. Show that the $C. G.$ of the figure so formed is on the line which bisects the hypotenuse and passes through the right-angle and divides it in the ratio 1: 26.

15. O is any point within the triangle ABC. Another triangle is formed by joining the $C. G.$ 'S G_1, G_2 and G_3 of triangles BOC, COA and AOB respectvely. Show that $\triangle G_1 G_2 G_3$ is similar to $\triangle ABC$ and one ninth of it.

16. ABC is a triangle. DE is a straight line drawn within the triangle, parallel to the base BC intersecting the other sides at D and E, DE and BC being equal to b and a respectively. If h be the length of the line segment drawn from A bisecting BC, prove that the distance of the $C.$ G. of the trapezium $BCED$ from A is

$$\frac{2}{3}\text{ h}\cdot\frac{a^2+ab+b^2}{a\,(a+b)}.$$

17. If the *C. G.* of a quadrilateral is the same as that of four equal particles placed at its angular points, prove that the quadrilateral must be a rectangle.

18. *ABCD* is a quadrilateral whose diagonals *AC* and *BD* intersect at *O*. If a point *E* be taken in *BD*, such that $BE=OD$, snow that the C. G. of $\triangle AEC$ is the same as that of the quadrilateral *ABCD*. [G.U. 1936]

19. A rigid framework, *ABCD*, of four equal rods forming part of a regular hexagon is suspended from *A*. Show that the angle made by *AB* with the vertical is

$$\tan^{-1}\left(\frac{4}{7}\sqrt{3}\right).$$

20. A rod of length $5a$ is bent so as to form five sides of a regular hexagon. Show that the distance of its C.G. from either end of the rod is

$$\frac{a}{10}\sqrt{133}\ .$$

21. Show that the C.G. of a quadrilateral is the same as that of four particles of equal weights placed at the four corners, together with a fifth particle of equal but negative weight placed at the intersection of the diagonals.

22. At each of $n-1$ of the angular points of a regular polygon of n sides, equal particles are placed. Show that the distance of the C.G. from the circum-centre of the polygon is $\frac{r}{n-1}$, where r is the circum-radius.

23. *ABCD* is trapezium in which *AB* and *CD* are parallel and of lengths *a* and *b* respectively. Prove that the distance of the C.G. of *ABCD* from the side *AB* is

$$\frac{h}{3}\cdot\frac{a+2b}{a+b},$$

h being the height of the trapezium. [C.U. 1944]

24. Weights proportional to 5, 4, 6, 2, 7 and 3 are placed at the angular points of a regular hexagon, taken in order. Show that their C.G. is the centre of the hexagon.

25. A triangle, *ABC*, of uniform wire has the side *CA* removed and is hung up by the point *A*. Show that for *BC* to be horizontal

$$b^2(c+2a)=c(c+a)^2,$$

where $BC=a$, $CA=b$ and $AB=c$. [C.U. 1941]

26. Show that the C.G. of a uniform triangular lamina is situated at the same point as that of three equal particles placed at the middle points of its sides.

27. The sides of a uniform triangular lamina are 5m, 6m and 7m respectively. Find the distances of its C.G. from the shortest and the longest sides.

$$\left[\text{Ans. } \frac{4}{5}\sqrt{6}\text{ m and } \frac{4}{7}\sqrt{6}\text{ m}\right]$$

28. If a triangular lamina, *ABC*, weighs 6 *N*, what weight must be placed at *A* so that the C.G. of the whole may bisect the line joining *A* to the middle point of *BC*?

[Ans. 2N]

29. The distances of the vertices of a triangular lamina from a straight line in its plane are x_1, x_2 and x_3. Find the distance of its C.G. from the line.

[C.U. 1947]

$$\left[\text{Ans. } \frac{1}{3}(x_1+x_2+x_3\right]$$

30. If three men support a heavy triangular board of weight *W* at its three corners, find the weight supported by each man.

$$\left[\text{Ans. } \frac{1}{3}\text{ w}\right]$$

31. A given weight is placed anywhere on a heavy uniform triangular lamina; show that the C.G. of the system lies within a certain triangle.

[C.U. 1927]

32. A uniform lamina in the shape of a right-angled triangle, and such that one of the sides containing the right angle is three times the other, is suspended by a string attached to the right angle. In the position of equilibrium, show that the hypotenuse is inclined at an angle $\sin^{-1}\frac{3}{5}$ to the vertical.

33. *ABC* is a triangular lamina and *D, E, F* are taken in *BC, CA* and *AB* such that

$$\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}.$$

Prove that the C.G. of the triangle *DEF* is the same as that of the triangle *ABC*.

34. A uniform triangular lamina *ABC* just topples over when placed in a vertical position with *BC* upon a smooth horizontal plane. Show that
$3 \sin A = \sin (B-C)$, $(B>C)$.

35. A uniform lamina in the shape of an isosceles triangle of vertical angle 2α is suspended from one extremity of the base. Show that, in the position of equilibrium, the base makes an angle
$\cot^{-}\left(\frac{1}{3}\cot \alpha\right)$ with the horizontal.

36. Masses proportional to $b+c$, $c+a$ and $a+b$ are placed at the points *A*, *B* and *C* of a triangular lamina, where a, b, c are the sides of the triangle. Show that their C.M. is at the in-centre of the triangle joining the middle points of the sides of the triangle *ABC*.

37. A triangular lamina having a right angle at *C* is suspended from the angle *A*, and the side *AC* makes an angle α with the vertical. It is then suspended from *B* and the sides *BC* makes an angle β with the vertical. Show that $\cot \alpha \cot \beta = 4$.

38. A triangular lamina, *ABC*, hangs at rest, one of the angular points *A* being supported at a fixed point. Prove that the angle which the lower side makes with the horizontal is $\tan^{-1}\left[\frac{1}{2}(\cot B - \cot C)\right]$

39. Prove that the C.G. of a uniform triangular lamina of mass *M*, bordered with a thin uniform rim of mass m, and loaded with a particle of mass $\frac{1}{2}$m at the in-centre is at the centroid of the triangle.

40. Find the C.G. of a uniform square plate, *ABCD*, of weight 10*N* together with weights of 20*N*, 30*N*, 40*N* and 50*N* placed at its four corners *A*, *B*, *C* and *D* respectively.

41. *G* is the C. G. of a uniform quadrilateral lamina, *G′* is the C.G. of four equal particles placed at its corners and *O* is the point of intersection of its diagonals. Prove that *O*, *G*, *G′* are in the same straight line and that $OG' = 3GG'$.

42. Show that the C.G. of a uniform quadrilateral lamina, *ABCD*, coincides with that of three particles of weights proportional to *AO*, *OC*, *2AC* placed at *A* and *C* and the mid-point of *BD*, where *O* is the point of intersection of *AC* and *BD*.

43. Three heavy particles are placed at the angular points A, B and C of a triangle, their weights being as $a:b:c$. Show that the distance of the C.G. of the particles from A is

$$\frac{2bc \cos \frac{A}{2}}{a+b+c}.$$

44. A solid hemisphere and a solid cylinder have the same radii and are made of the same homogeneous material, and one end of the cylinder is cemented to the base of the hemisphere. The height of the cylinder is $\frac{2}{3}$ of its radius. Show that the C.G. of the whole solid is $\frac{1}{48}$ of the radius from the common plane.

45. A cylinder and a cone have their bases jointed together, the bases being of the same size. Find the ratio of the height of the cone to the height of the cylinder so that the common C.G. may be at the centre of the common base.

46. A uniform solid figure consists of a right circular cone of semi-vertical angle α and a hemisphere of radius equal to that of the-base of the cone, two having a common flat base. Find the position of the C.G. and show that if the solid is suspended from a point on the rim of the base of the cone, the angle that the axis of the cone makes with the horizontal is

$$\tan^{-1}\left\{\frac{1-3\tan^2\alpha}{4\tan\alpha\,(1+2\tan\alpha)}\right\}.$$

[Ans. $\frac{(1-3\tan^2\alpha)\,\text{h}}{4(1+2\tan\alpha)}$ from the centre of the sphere, where h is the height of the cone.]

47. A uniform wire is bent into the form of an acute angled triangle, ABC, and AD is drawn perpendicular to BC. If the part BD of the wire is cut away, show that the distance of the C.G. of the remainder from BC is

$$\frac{c(b+c)\sin B}{2c+4b\cos^2\frac{C}{2}}.$$

48. A square, $ABCD$, is divided into four equal triangles by its diagonals which intersect at O. If the triangle OAB be removed, find G, the centroid of the remaining portion of the square.

[Ans. $OG=\frac{2}{9}\,OE$, where $OGE \perp CD$

49. G is the $C. G.$ of $\triangle ABC$. If $\triangle GBC$ be removed, find the distance of the C.G. of the remainder from A.

$\left[\text{Ans } \frac{5}{6} AG\right]$

50. If from a triangle ABC, three equal triangles ARQ, BPR and CQP be cut off, prove that the C.G.'s of the triangles ABC and PQR will be coincident.

51. A triangular portion is punched out from a square, whose two sides are adjacent sides of the square and the third is the join of the mid-points of these sides. Find the C.G. of the remaining portion.

$\left[\text{Ans. } \frac{1}{21} \times \text{ diagonal from the centre.}\right]$

52. ABC is a uniform triangular lamina. If the triangle formed by joining the mid-points of the sides of ABC be removed, prove that the C.G. remains unaltered.

53. The sides of a parallelogram, $ABCD$, are bisected and the points of bisection of the opposite sides are joined. If these lines meet at O, and if the small parallelogram AO be removed, find the C.G. of the remainder.

$\left[\text{Ans. On } OC \text{ at a distance } \frac{1}{6} OC \text{ from } O.\right]$

54. From a triangle is cut off $\frac{1}{9}$ th of its area by a straight line parallel to its base. Find the position of the C.G. of the remainder.

$\left[\text{Ans. Distance of } C.G. \text{ from } D = \frac{5}{18} AD, \text{ where } AD \text{ is a median of } \triangle ABC.\right]$

55. G is the point of intersection of the medians AD, BE and CF of $\triangle ABC$. If the portion $AFGE$ is removed, show that the C.G. of the remainder is on DG at a distance $\frac{7}{12} DG$ from D.

56. The middle points of two adjacent sides of a uniform triangular lamina are joined and the lamina is cut in two along the joining line. Find the C.G. of the larger portion. [C.U. 1942]

[Ans. The C.G. divides the line joining the middle points of the parallel sides in the ratio 4:5, being nearer the base.]

57. From a triangle, ABC, a portion ADE, where DE is parallel to BC, is removed. If a and b be the distances of A from BC and DE respeetively, show that the distance of the C.G. of the remainder from BC is [C.U. 1938]

$$\frac{a^2+ab-2b^2}{3(a+b)}.$$

58. From a thin uniform triangular board, *ABC*, the portion constituting the inscribed circle is removed. Prove that the distance of the C.G. of the remainder from the side *a* is

$$\frac{\triangle}{3as}\cdot\frac{2s^3-3\pi\, a\triangle}{s^2-\pi\triangle}.$$

$\triangle$ being the area and s, the semi-perimeter of the board.

59. Prove that if equal triangles be cut off from the corners of a given triangle by lines parallel to the respective opposite sides, the C.G. of the remainder will coincide with that of the triangle.

60. A uniform lamina is in the form of a square, *ABCD*, of side $2a$, from which the isosceles triangle, *ABE*, is cut away. In the triangle, $AE=BE$ and the distance of *E* from *AB* is h. Prove that the C.G. of the lamina is at a distance $\frac{12a^2-h^2}{3(4a-h)}$ from *AB*.

61. From a circular disc of radius 6 cm is cut a circle having a diameter equal to the radius of the disc. Find the C.G. of the remainder. [Ans. 1 cm from the centre of the circular disc along the line of centres away from the circle cut off.]

62. From a circular disc of radius r is cut out a circle which passes through the centre and whose diameter is $\frac{1}{3}$r. Find the C.G. of the remainder.

[C.U. 1940]

[Ans. At a distance $\frac{r}{210}$ from the centre of the disc on the line joining the centre of the disc to that of the hole, produced backwards.]

63. In a uniform circular disc of radius *R*, a circular hole of radius δr is punched out, the distance between the two centres being c, where $r+c<R$. Show that the C.G. of the remainder is at a distance $\frac{cr^2}{R^2-{}^2r}$ from the centre of the disc.

64. From a solid right circular cylinder, a solid right circular cone on the same base is scooped out. Find the ratio of the height of the cone to that of the cylinder if the C.G. of the remainder is at the vertex of the cone.

[Ans. $(2-\sqrt{2}):1$]

65. From a cone of height h and vertex A, the upper portion, a cone of height $\frac{h}{2}$ is removed. Show that the C.G. of the remainder is at a distance $\frac{45}{56}$ h from A.

66. A frustum of a cone is formed by cutting off the upper portion of a solid right circular cone by a plane parallel to the base. The radii of the parallel circular sections being R and r, and h, the height of the frustum, show that the height of the C.G. of the frustum from the base is

$$\frac{h}{4}\cdot\frac{R^2+2Rr+r^2}{R^2+Rr+r^2}.$$

67. A buoy is formed of a uniform thin sheet of metal in the form of a solid cone standing on a solid hemisphere with a common base. Find the vertical angle of the cone so that the combined C.G. may be at the centre of the hemisphere.

[Ans. 60°]

68. $ABCDEF$ is a regular hexagon, having O as the centre and a as the radius of its inscribed circle. If $\triangle AOB$ be removed, find the C.G. of the remaining area.

[Ans. $\frac{2}{15}$ a from O towards the middle point of DE.]

69. A pile of six rupees rests on a horizontal table and each rupee projects the same distance beyond one below it. Find the greatest possible horizontal distance between the centres of the highest and the lowest rupees. [P.U. 1937]

[Ans. $\frac{5}{3}$ r, where r is the radius of a rupee.]

70. Find the centroid of the area between the curve $y=\sin x$ and the line $y=0$, where $0 \leqslant x \leqslant \pi$

$$\left[\text{Ans. } \bar{x}=\frac{\pi}{2},\quad \bar{y}=\frac{\pi}{8}\right]$$

71. Find the centroid of the area bounded by $y=x^2$, $x=1$, $y=0$ and $x=2$.

$$\left[\text{Ans. } \bar{x}=\frac{45}{28},\quad \bar{y}=\frac{93}{70}\right]$$

72. Find the centroid of the area bounded by $y=x^2$, $y=-x^2$ and $x=3$.

$$\left[\text{Ans. } \bar{x}=\frac{9}{4},\ \bar{y}=0\right]$$

73. Find the centroid of the area bounded by $y^2=ax$, and $x^2=by$ $(a, b>0)$.

$$\left[\text{Ans. } \bar{x}=\frac{9}{20}\,a^{\frac{1}{3}}\,b^{\frac{2}{3}},\ \bar{y}=\frac{9}{20}\,a^{\frac{2}{3}}\,b^{\frac{1}{3}}\right]$$

74. Find the centroid of the area bounded by $x=0$, $y=0$ and $y=3+2x-x^2$ and lying in the positive quadrant.

$$\left[\text{Ans } \bar{x}=\frac{5}{4},\ \bar{y}=\frac{17}{10}\right]$$

Appendix

Some Multiple-Choice Objective Type Questions

Multiple-choice objective type questions are emphasized in most of the competitive examinations. A few such questions are, therefore given below from each topic for the convenience of students.

In each of the following questions, there are four possible answers marked (*A*). (*B*), (*C*) and (*D*). Only one of these answers is correct. The letter that has been given against the correct answer should be written in bold letter in the bracket provided in the right hand margin.

I. Topic : Fundamental concepts, Resultants and Components of Forces

1. The minimum requirement for specifying completely a force acting on a rigid body is the knowledge about its

   (A) magnitude, direction or point of application only
   (B) magnitude and direction or point of application only
   (C) magnitude, direction and point of application only
   (D) point of application and magnitude or direction only ( )

2. Let O be the middle point of the line segment $A'A$ and let $\vec{F}=\vec{OA}$, F being the magnitude of a force. Then $-\vec{F}$ is equal to

   (A) $-\vec{AO}$ which is again equal to $-\vec{OA'}$
   (B) $-\vec{AO}$ which is again equal to $\vec{A'O}$
   (C) $\vec{AO}$ which is again equal to $\vec{A'O}$
   (D) $\vec{AO}$ which is again equal to $\vec{OA'}$ ( )

3. Two forces $\vec{P}$ and $\vec{Q}$ are acting at a point O in the directions OA and OB respectively. Then the point O will be at rest if

(A) $OA=OB$

(B) $\vec{P}-\vec{Q}=0$

(C) $P=Q$

(D) $\vec{P}+\vec{Q}=0$ ( )

4. If a rod, AB, rests inside a smooth hollow sphere with centre at O, then the reactions of the sphere on the ends A and B of the rod will be along

(A) AO and BO respectively

(B) AB and BA respectively

(C) the tangents at A and B respectively

(D) any other direction ( )

5. If a rod, AB, rests upon a smooth peg or a support at O (or a smooth horizontal rail through O), then the reaction of the peg, the support or the rail at O on the rod will be along

(A) OB

(B) OA

(C) the normal to the rod at O above AB

(D) any other direction ( )

6. If the resultant of two forces, P and Q, acting at a point at an angle θ makes an angle α with Q, then $\tan \alpha$ is equal to

(A) $\dfrac{P \sin \theta}{Q+P \cos \theta}$

(B) $\dfrac{Q \sin \theta}{P+Q \cos \theta}$

(C) $\dfrac{Q \cos \theta}{Q+P \sin \theta}$

(D) $\dfrac{Q \cos \theta}{P+Q \sin \theta}$ ( )

7. If R be the resultant of two forces, P and Q, acting at a point making an angle α with each other and if β be the angle between P and R, then $P : Q : R$ is equal to

(A) $\sin\alpha : \sin\beta : \sin(\alpha-\beta)$
(B) $\sin\beta : \sin\alpha : \sin(\alpha+\beta)$
(C) $\sin(\alpha+\beta) : \sin\beta : \sin\alpha$
(D) $\sin(\alpha-\beta) : \sin\beta : \sin\alpha$ ( )

8. If two equal forces each of magnitude P act at an angle θ, then their resultant will be

(A) $\frac{P}{2}\cos\frac{\theta}{2}$
(B) $2P\cos\frac{\theta}{2}$
(C) $2P\sin\frac{\theta}{2}$
(D) $2P\cos\theta$ ( )

9. The resolved part of the resultant of two forces inclined at an angle θ, in a given direction, is equal to

(A) the algebraic sum of the resolved parts of the forces in the same direction
(B) the arithmetic sum of the resolved parts of the forces in the same direction
(C) the difference of the forces multiplied by $\cos\theta$
(D) the sum of the forces multiplied by $\sin\theta$ ( )

10. According to the principle of transmissibility of forces, the effect of a force upon a body is

(A) maximum when it acts at the C.G. of the body
(B) different at different points in its line of action
(C) the same at every point in its line of action
(D) minimum when it acts at the C.G. of the body ( )

11. The resultant of two forces of magnitudes $5N$ and $10N$ acting at a point can never be equal to

(A) $10N$
(B) $5N$
(C) $8N$
(D) $4N$ ( )

12. The sum of the two forces is $18N$ and the resultant whose direction is at right angles to the smaller force is $12N$. Then the magnitude of the forces are
    (A) $11N$ and $7N$
    (B) $13N$ and $5N$
    (C) $10N$ and $8N$
    (D) none or the above
    ( )

13. The resultant of two forces, P and Q, is $\sqrt{3}\,Q$ and makes an angle of 30° with the direction of P. Then either
    (A) $P=Q$ or $P=2Q$
    (B) $P=Q$ or $2P=Q$
    (C) $2P=Q$ or $P=2Q$
    (D) none of the above
    ( )

14. The resultant of two equal forces, P, P, acting at a point and inclined at an angle of 120° with each other makes an angle θ with each of the equal forces. Then θ is
    (A) 0°
    (B) 30°
    (C) 60°
    (D) 90°
    ( )

15. Two forces whose magnitudes are in the ration 3 : 4 act at a point at right angles to each other. If their resultant is $75N$, then the forces are
    (A) $45N$, $60N$
    (B) $30N$, $40N$
    (C) $24N$, $32N$
    (D) none of the above
    ( )

16. Ox and Oy are two perpendicular lines which intersect at O. A force $\vec{F}$ makes an angle $\pi-\theta$ with Oy. Then the resolved part of $\vec{F}$ along Ox is
    (A) $-\vec{F}\cos\theta$
    (B) $\vec{F}\cos\theta$
    (C) $\vec{F}\sin\theta$
    (D) $-\vec{F}\sin\theta$
    ( )

17. Two forces, P and Q, act at a point O and are inclined to each other at an angle α. If the resultant force, R, makes an angle θ with the force, P, then $\sin\theta$ is equal to

(A) $\dfrac{Q\sin\alpha}{P}$

(B) $\dfrac{R\sin\alpha}{Q}$

(C) $\dfrac{R\sin\alpha}{P}$

(D) $\dfrac{Q\sin\alpha}{R}$ ( )

18. If the forces, P and $2P$, act at O and are inclined to each other at an angle of 120,° then the resultant of the forces makes an angle θ with the force $2P$, where θ is equal to

(A) 30°
(B) 60°
(C) 90°
(D) none of the above ( )

19. $ABCD$ is a square and forces acting at a point are represented in magnitude and direction by AB, $2BC$, $2CD$ and DA. Then their resultant is represented in magnitude and direction by

(A) 4 AD
(B) 4 BA
(C) BD
(D) AC ( )

20. If forces $\vec{R}$, $\vec{F_1}$ and $\vec{F_2}$ are represented in magnitude and direction by $\vec{CA}$, $\vec{AB}$ and $\vec{BC}$ respectively of $\triangle ABC$, then

(A) $\vec{R}=\vec{F_1}+\vec{F_2}$

(B) $\vec{R}=\vec{F_1}-\vec{F_2}$

(C) $\vec{R}=\vec{F_2}-\vec{F_1}$ ( )

(D) $\vec{R}=-\vec{F_1}-\vec{F_2}$

21. If a force, P, be resolved into two components making angles 45° and 15° with its direction, then the latter force is

(A) $\sqrt{2}P$

(B) $\dfrac{P}{\sqrt{6}}$

(C) $\dfrac{\sqrt{6}P}{3}$

(D) none of the above ( )

22. Forces of magnitudes $3N$, $4N$, $5N$ and $6N$ act at a point in the directions North, South, East and West respectively. Then the magnitude of their resultant is

(A) $\sqrt{2}N$

(B) $18N$

(C) $2N$

(D) none of the above ( )

23. If R be the resultant of two forces, P and Q, inclined at an angle α with each other and if θ be the angle between Q and R, then $\tan\theta$ is equal to

(A) $\dfrac{2P\tan\frac{\alpha}{2}}{(Q-P)\sec^2\frac{\alpha}{2}+2P}$

(B) $\dfrac{2Q\tan\frac{\alpha}{2}}{(P-Q)\sec^2\frac{\alpha}{2}+2Q}$

(C) $\dfrac{2P\tan\frac{\alpha}{2}}{(P-Q)\sec^2\frac{\alpha}{2}+2Q}$

(D) none of the above ( )

24. In problem no. 23, the resultant R is given by the equation

(A) $R^2-2PQ\ \sin^2\frac{\alpha}{2}=(P+Q)^2$

(B) $R^2+2PQ\ \sin^2\frac{\alpha}{2}=(P-Q)^2$

(C) $R^2+2PQ\ \sin^2\frac{\alpha}{2}=(P+Q)^2$

(D) none of the above ( )

25. The resultant of two forces whose magnitudes are $12N$ and $8N$ will be greatest when they act in

(A) opposite directions
(B) perpendicular directions
(C) the same direction
(D) any other direction ( )

26. Two forces $\overrightarrow{OA}$ and $\overrightarrow{OB}$ act at O. If C be the mid-point of AB, then the resultant of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is

(A) $\overrightarrow{OC}$

(B) $2\overrightarrow{OC}$

(C) $\overrightarrow{CO}$
(D) unpredictable ( )

II. Topics : Equilibrim of Concurrent Forces

1. Three forces act at a point. Then the forces will always be in equilibrium if

(A) they can be represented in magnitude and direction by the three sides of a triangle, taken in any arbitrary order
(B) they can be represented in magnitude and direction by the three sides of a triangle, taken in a definite order
(C) they are parallel to the sides of a traingle, taken in a definite order
(D) their magnitudes are proportional to the sides of a triangle, taken in a definite order ( )

2. A number of forces acting at a point will be always in equilibrium if
   (A) their algebraic sum is zero
   (B) the algebraic sums of their resolved parts in two mutually perpendicular directions are equal
   (C) the algebraic sums of the resolved parts of the forces in any two mutually perpendicular directions are separately zero
   (D) the forces are equally inclined to each other ( )
3. Three forces act at a point. The set of these three forces which are never in equilibrium is
   (A) $3N$, $5N$ and $9N$
   (B) $8N$, $6N$ and $4N$
   (C) $13N$, $31N$ and $35N$
   (D) $10N$, $20N$ and $25N$ ( )
4. A body of weight $30N$ is suspended by two strings of lengths 3m and 4m attached to two points in the same horizontal line whose distance apart is 5m. Then the tensions of the strings are
   (A) $42N$, $18N$
   (B) $24N$, $18N$
   (C) $50N$, $37.5\ N$
   (D) none of the above ( )
5. A body of weight $20N$ is placed on a smooth plane of inclination 60° to the horizon and a force is acting at a point of the body parallel to the surface along the line of the greatest slope such that the body is in equilibrium.
   Then the force is
   (A) $10\sqrt{3}\,N$
   (B) $3\sqrt{10}\,N$
   (C) $\sqrt{103}\,N$
   (D) none of the above ( )
6. Two forces, each equal to P, act at right angles to each other. Their effect is neutralized by a third force acting along the internal bisector of the angle between the forces in the opposite directions. Then the third force is
   (A) $2P$
   (B) $\frac{1}{2}P$
   (C) $\sqrt{2}P$
   (D) $\frac{1}{\sqrt{2}}P$ ( )

7. Three forces acting on a particle are in equilibrium and are proportional to $\sqrt{3}+1$, $\sqrt{6}$ and 2. The angles at which they are inclined to each other are

(A) 105°, 102° and 153°
(B) 105°, 120° and 135°
(C) 150°, 102° and 108°
(D) none of the above ( )

8. A body of weight W Newton is suspended from a point C by two strings, AC and BC, of lengths 3m and 4m, whose other ends are fastened to two points, A and B, in the same horizontal line. O is the point of intersection of the vertical through A and BC produced. If $AB=5$m and T_1, T_2 be the tensions of the strings CA and CB respectively, then $T_1 : T_2 :$ W is equal to

(A) $CO : CA : AO$
(B) $AO : CA : CO$
(C) $CA : OC : AO$ ( )
(D) none of the above

9. In problem no. 8, if W$=5N$, then T_1 and T_2 will be respectively equal to

(A) $4N$ and $3N$
(B) $3N$ and $4N$
(C) $5N$ and $4N$
(D) $5N$ and $3N$ ( )

10. Three forces acting on a particle can never be kept in equilibrium if they are of magnitudes

(A) $\sqrt{7}\,N$, $\sqrt{10}\,N$ and $\sqrt{15}\,N$
(B) $3\sqrt{7}\,N$, $7\sqrt{3}\,N$ and $\sqrt{21}\,N$
(C) $2\sqrt{3}\,N$, $3\sqrt{2}\,N$ and $\sqrt{17}\,N$
(D) $\sqrt{7}\,N$, $\sqrt{17}\,N$ and $\sqrt{71}\,N$ ( )

11. Three forces acting at a point are always in equilibrium if they are represented in magnitude and direction by

(A) the three altitudes of a triangle, taken in order
(B) the three sides of a triangle, taken in order
(C) the three medians of a triangle, taken in order
(D) the three bisectors of the angles of a triangle, taken in order ( )

12. A, B, C and D, taken in order, are any four points on a circle with centre O such that $\angle AOB=\theta_1$, $\angle BOC=\theta_2$, $\angle COD=\theta_3$ and $\angle DOA=\theta_4$. Four forces, each of magnitude equal to the radius of the circle, act along OA, OB, OC and OD. Then the four forces will be always in equilibrium if

(A) $\theta_1=\theta_2$
(B) $\theta_4=\theta_1$
(C) $\theta_3=\theta_1$
(D) $\theta_3=\theta_4$ ( )

13. Three sets of of concurrent forces act at a point O. The first set of forces consists of two forces, $3N$ and $4N$, acting at right angles. The second set consists of forces, $4N$, $5N$ and $6N$, acting in the directions indicated by the sides AB, BC and CD of the triangle ABC with sides $AB=2$m, $BC=2.5$m and $CD=3$m. The third set consists of forces $4N$, $3N$, $6N$ and $7N$ acting in the directions indicated by the sides PQ, QR, RS and SP of the quadrilateral $PQRS$ with sides equal to the magnitude of the forces. Then the resultant of all these forces is

(A) 42
(B) 0
(C) 5
(D) none of the above ( )

14. Three forces of magnitudes $3\ N$, $\sqrt{12}\ N$ and $\sqrt{3}\ N$ act at a point O towards the point O and if these forces are in equilibrium, then the angles between P,Q ; Q,R and R,P will be respectively

(A) 30°, 60°, 90°
(B) 90°, 60°, 30°
(C) 60°, 30°, 90°
(D) 30°, 90°, 60° ( )

15. A weight, W, is supported by two cables making angles 60° and θ (which is actue) with the horizontal. Then the tension in the cable making angle θ with the horizontal will be minimum when θ is equal to

(A) 45°
(B) 60°
(C) 30°
(D) none of the above ( )

16. A chord, ABC, is attached at two points, A and C, on two vertical walls. A pulley, B, of infinitesimally small radius carries a load, W, and is free to roll without friction along the chord. If T_1 and T_2 be the tensions of the two portions, AB and BC, of the string respectively, then
   (A) $T_1 > T_2$
   (B) $T_1 = T_2$
   (C) $T_1 < T_2$
   (D) the tensions are unpredictable ( )

17. If in problem no. 16, AB and BC make angles α and β respectively with the horizontal, then
   (A) $\alpha = \beta$
   (B) $\alpha > \beta$
   (C) $\alpha < \beta$
   (D) α and β are unpredictable ( )

18. Three forces P, Q, R act at a point and are in equilibrium. The angle between R and Q is α and that between P and R is β. Let $P+Q+R=K$ and $\operatorname{cosec}\alpha+\operatorname{cosec}\beta+\cot\alpha+\cot\beta=l$. Then the forces P, Q and R are respectively equal to
   (A) $\dfrac{K\operatorname{cosec}\beta}{l}, \quad \dfrac{K\operatorname{cosec}\alpha}{l}, \quad \dfrac{K(\cot\alpha+\cot\beta)}{l}$
   (B) $\dfrac{K\operatorname{cosec}\alpha}{l}, \quad \dfrac{K\operatorname{cosec}\beta}{l}, \quad \dfrac{K(\cot\alpha+\cot\beta)}{l}$
   (C) $\dfrac{K\operatorname{cosec}\beta}{l}, \quad \dfrac{K\operatorname{cosec}\alpha}{l}, \quad \dfrac{K(\cot\alpha-\cot\beta)}{l}$
   (D) none of the above ( )

III. Topics : Parallel Forces, Moments and Couples

1. If three parallel forces are in equilibrium, the magnitude of each force is proportional to
   (A) the distance between the other two
   (B) the square of the distance between the other two
   (C) the inverse distance between the other two
   (D) none of the above ( )

2. Parallel forces P, Q, R act at the vertices A, B, C respectively of a triangle ABC and are such that $P : Q : R = \tan A : \tan B : \tan C$. Then their resultant will pass through the
   (A) in-centre
   (B) centroid
   (C) otho-centre
   (D) circum-centre ( )

3. A man carries a bundle of weight W units, hung at one end of a stick of length l units, which is placed over his shoulder. If the distance x units between his hand and shoulder be altered, then the pressure P units on his shoulder will change such that
   (A) $P \propto x$
   (B) $P \propto \frac{1}{x}$
   (C) $P \propto x^2$
   (D) none of the above is true ( )

4. Two men carry a uniform bar, 14m long and weighing 154N. One man supports it at a distance of 1 m from one end and the other man at a distance of 5m from the middle point of the bar. Then the weights which the two men bear are
   (A) 70N and 84N
   (B) 48N and 70N
   (C) 80N and 14N
   (D) none of the above ( )

5. A straight uniform rod is 1 m long. When a load of xN is placed at one end, it balances about a point 10 cm from that end. Then the weight of the rod is
   (A) $\frac{x}{4} N$
   (B) $2x\ N$
   (C) $\frac{x}{2} N$
   (D) none of the above ( )

6. Three equal like parallel forces act at the vertices of a triangle. Then their resultant passes through
   (A) the in-centre of the triangle
   (B) the circum-centre of the triangle
   (C) the ortho-centre of the triangle
   (D) the centroid of the triangle ( )

7. A rod weighing W units and of length 1m is placed on a table so that one-third of its length projects over the edge. The greatest weight which can be attached by a string to the end of the rod without causing it to topple over is

(A) $4W$ nnits
(B) $6W$ units
(C) $2W$ units
(D) none of the above ( )

8. Varignon's theorem is about finding

(A) the algebraic sum of the resolved parts of forces
(B) the algebraic sum of the moments of forces
(C) the conditions of equilibrium of forces
(D) none of the above ( )

9. The only case when we can not find the resultant force of a number of coplanar forces acting on a body is when the forces

(A) act at diflerent points in different directions
(B) reduce to two like parallel forces
(C) reduce to two unlike and unequal parallel forces
(D) reduce to a couple only ( )

10. The moment of a force, F, about a point O at a distance d from the line of action of the force can not be zero if :

(A) $F=0$, but $d\neq0$
(B) $d=0$, but $F\neq0$
(C) $d=0$ and $F=0$
(D) $d\neq0$ and $F\neq0$ ( )

11. A single force and a couple acting in the same plane upon a rigid body

(A) can produce equilibrium of the body
(B) are equivalent to a single force
(C) are equivalent to a couple
(D) are equivalent to each other ( )

12. If three forces acting upon a rigid body be represented in magnitude, direction and line of action by the sides of a triangle, taken in order, then they are

(A) equivalent to a couple whose moment is equal to twice the area of the triangle

(B) in equilibrium

(C) equivalent to a single resultant

(D) equivalent to a couple whose moment is equal to the area of the triangle ( )

13. The resultant of a force 7 *N* and a couple in the same plane, whose arm is 3.5 *m* and whose forces are each 4 *N* is

(A) 7 *N*

(B) 7 *Nm*

(C) 14 *N*

(D) 4 *Nm* ( )

14. Three forces always form a couple if

(A) they act along the sides of a triangle in any direction, whatever be their magnitudes

(B) they are represented in magnitude and direction by the sides of a triangle, taken in order

(C) they act along the sides of a triangle and are represented in direction only by the sides of a triangle, taken in order

(D) they are represented in magnitude and direction by the sides of a triangle, taken in order, and they act along the sides of the triangle ( )

15. Two coplanar couples are said to be equivalent when

(A) their moments are equal in magnitude and sign

(B) their moments are equal in magnitude but opposite in sign

(C) their arms are equal whatever be their forces

(D) their forces are equal in magnitude and direction whatever be their arms ( )

16. If the forces 6 W, 5 W acting at a point (2, 3) in Cartesian rectangular co-ordinate plane are parallel to the positive direction of x and y axes respectively, then the algebraic sum of the moments of the forces about the origin is

(A) 11 W
(B) -3 W
(C) -8 W
(D) 3 W ( )

17. Four forces are represented in magnituted and direction by the sides of a quadrilateral, $ABCD$, taken in order. Then the four forces

(A) are in equilibrium
(B) can be reduced to two like parallel forces
(C) can be reduced to two unequal unlike parallel forces
(D) are equivalent to a couple ( )

18. The force, arm and sense of each of three couples are as follows :

(i) 20 N, .5 m, positive sense
(ii) 30 N, 1 m, negative sense
(iii) 40 N, .25 m, positive sense

Then the force and the sense of the resultant of these three couples with arm .5 m will be

(A) 20 N, negative sense
(B) 20 N, positive sense
(C) 10 N, positive sense
(D) 10 N, positive sense ( )

19. The resultant of two unlike parallel forces, P and Q ($P>Q$), acting at points, A and B, respectively of the body acts at a point C in AB. Then

(A) C bisects AB
(B) C divides AB internally and $\frac{CB}{CA}=\frac{P}{Q}$
(C) C divides AB externally and $\frac{CB}{CA}=\frac{P}{Q}$
(D) C divides AB externally and $\frac{CB}{CA}=\frac{Q}{P}$ ( )

IV. Topics : Reduction of Coplanar forces, Equilibrium of a Rigid body and Centre of Gravity

1. The end A of a heavy uniform rod, AB, is hinged with a vertical wall. The other end B of the rod is tied with a string whose other end is attached with the wall at a point D, vertically above A. The vertical through G, the middle point of AB, intersects BD at O. Then the rod will remain in equilibrium if the reaction at the hinge
   (A) is perpendicular to the wall
   (B) is perpendicular to the rod
   (C) passes through G
   (D) passes through O ( )

2. A heavy uniform rod, AB, rests with the end A against a smooth vertical wall and the other end B is tied to a point C of the wall, vertically above A by a string, BC. If the vertical through G, the middle point of AB, cuts BC at O, then the rod will remain in equilibrium if the reaction of the wall on the rod at A
   (A) passes through O and makes any arbitrary angle with the wall
   (B) passes through O and makes an angle 90° with the wall
   (C) is perpendicular to the wall but does not pass through O
   (D) is perpendicular to the rod and passes through O ( )

3. A heavy uniform rod, AB, is hung from a point O with two strings, AO and BO, of unequal lengths, attached at the points A and B of the rod. If the rod remains in equilibrium, then the weight of the rod
   (A) must pass through O and will be equally inclined with AO and BO
   (B) must pass through O and will be equally inclined with AB
   (C) must pass through O but can not make equal angles with AB
   (D) can not pass through O at all ( )

4. A heavy uniform rod, AB, is hung from a smooth nail at O by an inextensible light string which passes over the nail and whose two ends are attached with the two ends A,B of the rod. If the length of the string is twice the length of the rod, then for equilibrium of the rod, the triangle OAB must be
   (A) isosceles, but not right-angled
   (B) right-angled isosceles
   (C) equilateral
   (D) scalene ( )

5. OA and OB are the lines of greatest slope of two smooth inclined planes and these two lines make angles of 30° and 45° respectively with the horizontal ground at O. A ball is resting on OA and OB. Then the reaction of the plane, OA, on the ball will be
   (A) greater than that on OB
   (B) less than that on OB
   (C) equal to that on OB
   (D) unpredictable ( )

6. A heavy ball rests on a smooth ground touching against a smooth vertical wall. The ball is pressed against the wall by a string OC, O being the centre of the ball and C, the point on the wall above the point of contact of the ball with the wall. If R_H and R_V be respectively the horizontal reactions of the wall and the vertical reaction of the ground on the ball and T, W be respectively the tensions of the string and the weight of the ball, then
   (A) $R_H = R_V = T = W$
   (B) $R_V = 0$, $R_H = W$, $T = 0$
   (C) $R_H = R_V = W$, $T = 0$
   (D) none of the above is true ( )

7. A heavy uniform rod of weight W is freely jointed at A and B, not in the same horizontal line. Then the only possibility of the rod AB being in equilibrium is that the lines of reactions at the joints will
   (A) cut the line of action of W at two different points
   (B) cut the line of action of W at the same point
   (C) act along the rod AB
   (D) either cut the line of action of W at the same point or be vertical ( )

8. In problem 7, if the rod, AB, is vertical, then, for equilibrium, the lines of reactions at the joints will be
   (A) horizontal
   (B) vertical
   (C) zero ( )
   (D) none of the above

9. In problem 7, if the rod, AB, is horizontal, then, for equilibrium, the lines of reactions at the joints will be
   (A) vertical
   (B) horizontal
   (C) zero ( )
   (D) none of the above

10. In problem 7, if the rod, *AB*, is light i.e. its weight is negligible, then for all positions of the rod *AB*, the directions of the lines of reactions at the joints for which the rod remains in equilibrium must be

(A) horizontal
(B) vertical
(C) along the rod
(D) different for different positions of the rod ( )

11. One end of a light rod, *AB*, is hinged at *A* with a vertical wall and the other end *B* is attached with a string, *BC*, *C* being a point on the wall vertically above *A*. If a weight *W* is attached at the end *B* of the rod so that the rod remains in equilibrium in horizontal position and if $\angle ABC=\theta$, then the force induced in the string *BC* is

(A) $W \operatorname{cosec} \theta$
(B) $W \sec \theta$
(C) $W \tan \theta$
(D) $W \cot \theta$ ( )

12. The direction of the reaction at the hinge at *A* in problem no. 11 will be

(A) along *AC*
(B) along *AB*
(C) along *BA*
(D) along any other direction ( )

13. The magnitude of the reaction of the hinge at *A* in problem no. 11 will be

(A) $W \operatorname{cosec} \theta$
(B) $W \sec \theta$
(C) $W \tan \theta$
(D) $W \cot \theta$ ( )

14. A heavy non-uniform bar, AB, is resting inside a smooth hollow hemisphere with its ends in contact with the inner surface of the hemisphere. Then the reactions of the hemisphere at the ends A and B of the rod will act
(A) horizontally
(B) vertically
(C) along AB
(D) along the normals at A and B to the surface ( )

15. In problem no. 14, the rod, AB, can remain in equilibrium only when
(A) it is horizontal
(B) it is not horizontal
(C) it makes an angle of 45° with the horizontal
(D) it makes an angle of 60° with the horizontal ( )

16. In problem no. 14, if the rod, AB, is uniform, then the only position of equilibrium of the rod is when
(A) it is horizontal
(B) it is not horizontal
(C) it makes an angle of 45° with the horizontal
(D) it makes an angle of 60° with the horizontal ( )

17. In problem no. 14, if the rod is uniform and is of weight W and length a, the radius of the hemisphere, then the reactions R_1 and R of the hemisphere on the rod at A and B respectively will be given by
(A) $R_1=W\sqrt{3},\ R_2=\dfrac{W}{\sqrt{3}}$
(B) $R_1=R_2=W$
(C) $R_1=R_2=\dfrac{W}{\sqrt{3}}$
(D) none of the above ( )

18. A rigid body acted on by two forces can be in equilibrium if and only if
(A) the two forces are equal in magnitude whatever be their directions
(B) the two forces are equal in magnitude, collinear and act in opposite directions
(C) the forces are equal, unlike and parallel
(D) the two forces act along the same line in opposite directions ( )

19. The necessary and sufficient conditions for the equilibrium of a system of coplanar forces acting on a rigid body are that

(A) all the forces of the system should be equal in magnitude
(B) the resultant of all the forces should vanish
(C) the resultant couple should vanish
(D) the resultant force and the resultant couple should vanish ( )

20. Three forces of magnitudes $2N$, $3N$ and $4N$ act along the sides AB, BC and CA respectively of a triangle ABC. If $AB=4m$, $BC=6m$ and $CA=8m$, then these three forces are equivalent to

(A) a force of magnitude 0
(B) a force of magnitude $9N$
(C) a force of magnitude $9N$ and a couple
(D) a couple only ( )

21. The algebraic sum of the moments of a system of coplanar forces, not in equilibrium, is zero about each of two points, A and B. Then the algebraic sum of the resolved parts of the force system in the direction perpendicular to AB is

(A) non-zero other than AB
(B) AB
(C) 0
(D) indeterminate ( )

22. A system of coplanar forces acts on a rigid body. The algebraic sum of the resolved parts of the forces about x and y-axes in the plane of the forces are X and Y respectively and the algebraic sum of the moments of all the forces about the origin of co-ordinates is G. If $X=2$, $Y=3$ and $G=8$, then the line of action of the forces passes through the point whose co-ordinates are

(A) $(0, -4)$
(B) $(0, 4)$
(C) $\left(0, -\frac{8}{3}\right)$
(D) none of the above ( )

23. A system of coplanar forces acting on a rigid body, not in equilibrium, can never be reduced to

    (A) two forces acting through two given points

    (B) two forces, one of which acts through a given point and the other along a given straight line

    (C) three forces acting at a point, which are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards

    (D) three forces acting along the sides of a given triangle in the same plane ( )

24. The upper end B of a ladder of weight W rests against a smooth vertical wall and the other end A rests on a rough horizontal ground. If the ladder makes an angle 60° with the horizontal, then the total reaction of the ground on the ladder at A is

    (A) W

    (B) $\frac{W}{2}$

    (C) more than W

    (D) none of the above ( )

25. A heavy ladder resting on a floor and against a vertical wall may not be in equilibrium if

    (A) both the floor and the wall are rough

    (B) both the floor and the wall are smooth

    (C) the floor is smooth, but the wall is rough

    (D) the floor is rough, but the wall is smooth ( )

26. A hollow metal ball filled with air and having a diameter 10 m and total mass 50 kg is fixed to one end of a uniform rod of length 50m and mass 100 kg Then the ball and the rod will balance horizontally about a point on the rod, at a distance

    (A) 20 m from the centre of the rod

    (B) 10 m from the centre of the rod

    (C) 25 m from the centre of the rod

    (D) 50 m from the centre of the rod ( )

27. The centre ef gravity of a solid cone of height h lies on the axis at a height h_1 above the base where h_1 is equal to

(A) $\frac{1}{4}h$

(B) $\frac{1}{3}h$

(C) $\frac{1}{2}h$

(D) $\frac{3}{8}h$

( )

28. The centre of gravity of a thin hollow cone of height h lies on its axis at a height h_1 above the base where h_1 is equal to

(A) $\frac{1}{4}h$

(B) $\frac{1}{3}h$

(C) $\frac{1}{2}h$

(D) $\frac{3}{8}h$

( )

29. The centre of gravity of a solid hemisphere of radius r lies on its central radius at a distance h from its plane base, where h is equal to

(A) $\frac{3\,r}{2}$

(B) $\frac{3\,r}{4}$

(C) $\frac{r}{2}$

(D) $\frac{3\,r}{8}$

( )

30. The centre of gravity of a plane lamina will not be at its geometrical centre in the case of a

(A) right-angled triangle

(B) equilateral triangle

(C) square

(D) circle

( )

31. From a circular plate of diameter 6 *cm* is cut out a circle whose diameter is equal to the radius of the plate. Then the distance of the C. G. of the remainder from the centre of the circular plate is

(A) 1.0 *cm*
(B) 1.5 *cm*
(C) .5 *cm*
(D) none of the above ( )

32. The centre of gravity of a trapezium lies

(A) on the line joining the mid-points of the parallel sides
(B) on the line joining the mid-points of the oblique sides
(C) at the intersection of its diagonals
(D) on some line or at some point other than above ( )

33. A pile of six circular plates rests on a horizontal table and each plate projects the same distance beyond one below it. If r be the radius of each plate, then the greatest possible horizontal distance between the centres of the highest and the lowest plates is

(A) $3r$
(B) $\frac{5r}{2}$
(C) $\frac{5r}{3}$
(D) none of the above ( )

34. A light rod, *AB*, is free to turn in a vertical plane about *A* which is attached to a vertical wall. The end *B* of the rod is attached with a light string, *BC*, where *C* is a point on the wall vertically above *A*. A weight 1500 *N* is now suspended from the end *B* and the system remains in equilibrium. If the $AB=5$m, $BC=3$m and $AC=6$m, then the tension of the string and the reaction at the joint *A* are respectively

(A) 1500 *N* and 1500 *N*
(B) 750 *N* and 1250 *N*
(C) 1250 *N* and 750 *N*
(D) none of the above ( )

35. The centre of gravity of a triangular lamina coincides with that of three equal weights placed at

(A) the angular points of the triangle

(B) the mid-points of the sides of the triangle

(C) the in-centre, the circum-centre and the ortho-centre of the triangle

(D) the in-centre, the circum-centre and the centroid of the triangle ( )

36. The centroid of the area bounded by $y=x^2$, $x=1$, $x=2$ and $y=0$ lies on the line

(A) $y=0$

(B) $x=1\frac{1}{2}$

(C) $y=\frac{1}{2}$

(D) other than above ( )

37. The co-ordinates of the centroid of the area bounded by $y=x^2$ $y=-x^2$ and $x=3$ will satisfy the equation

(A) $x=2$

(B) $x=2\frac{1}{4}$

(C) $x=0$

(D) $x+y=0$ ( )

Answers to multiple-choice objective type questions

I. *1.* (C) *2.* (D) *3.* (B) *4.* (A) *5.* (C) *6.* (A) *7.* (D) *8.* (B) *9.* (A) *10.* (C) *11.* (D) *12.* (B) *13.* (A) *14.* (C) *15.* (D) *16.* (C) *17.* (D) *18.* (A) *19.* (C) *20.* (D) *21.* (C) *22.* (A) *23.* (A) *24.* (C) *25.* (C) *26.* (B)

II. *1.* (B) *2.* (C) *3.* (A) *4.* (B) *5.* (A) *6.* (C) *7.* (B) *8.* (C) *9.* (A) *10.* (D) *11.* (B) *12.* (C) *13.* (C) *14.* (A) *15.* (C) *16.* (B) *17.* (A) *18.* (A)

III. *1.* (A) *2.* (C) *3.* (B) *4.* (A) *5.* (A) *6.* (D) *7.* (C) *8.* (B) *9.* (D) *10.* (D) *11.* (B) *12.* (A) *13.* (B) *14.* (D) *15.* (A) *16.* (C) *17.* (D) *18.* (A) *19.* (C)

IV. *1.* (D) *2.* (B) *3.* (C) *4.* (C) *5.* (A) *6.* (D) *7.* (D) *8.* (B) *9.* (A) *10.* (C) *11.* (A) *12.* (B) *13.* (D) *14.* (D) *15.* (B) *16.* (A) *17.* (C) *18.* (B) *19.* (D) *20.* (D) *21.* (C) *22.* (A) *23.* (C) *24.* (C) *25.* (B) *26.* (B) *27.* (A) *28.* (B) *29.* (D) *30.* (A) *31.* (C) *32.* (A) *33.* (C) *34.* (B) *35.* (A) *36.* (D) *37.* (B)

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